MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL

MAY 2012

SUBJECT:	APPLIED MATHEMATICS
PAPER NUMBER:	I
DATE:	15th May 2012
TIME:	9.00 a.m. to 12.00 noon

Directions to candidates

Attempt all questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. In the triangle ABC, the side AB = $4a$, AC = $3a$ and the angle BAC = 90°. Forces of magnitude $17P$, $15P$, $3P$ act along AB, BC and AC respectively in the directions implied by the order of the letters.

   (i) Find the magnitude of the resultant of these forces, and the tangent of the angle made by its line of action with AB.

   (ii) Find the distance from A of the point where the line of action of the resultant cuts AB.

   (iii) A couple G is now added to the system and the resultant of this enlarged system acts through the point B. Calculate the magnitude and sense of G.

[5, 3, 2 marks]

2. A particle is uniformly accelerated from A to B, a distance of 192 m, and is then uniformly retarded from B to C, a distance of 60 m. The speeds of the particle at A, B and C, which lie in a straight line, are 4 ms^{-1}, V ms^{-1}, and 0 ms^{-1} respectively. The total time taken by the particle to move from A to C is 22 s.

   (i) Draw a velocity-time diagram for this motion.

   (ii) Express, in terms of V only, the times taken by the particle to move from A to B, and from B to C.

   (iii) Find the value of V.

   (iv) Find the acceleration of the particle between A and B.

[2, 2, 5, 1 marks]

3. A uniform right circular solid cylinder has radius r and length $4r$. A solid hemisphere of radius r is cut from one end of the cylinder, the plane face of the hemisphere being one of the end faces of the cylinder. The hemisphere so removed, is now attached by its plane face to the uncut plane face of the cylinder, thus forming a new solid.

   (i) Find the position of the centroid of this solid.

   (ii) P is a point on the rim of the solid on the side where the hemisphere was removed. The solid is suspended smoothly from P, and hangs freely. Find the angle which the axis of the solid makes with the vertical.

   Hint: The centroid of a uniform hemisphere of radius r is at a distance of $\frac{3r}{8}$ from the centre.

[7, 3 marks]

4. A truck of mass 6000 kg is moving with a speed of 12 ms^{-1} along a straight horizontal railway line when it collides with another truck B of mass 9000 kg which is stationary. After the collision, the two trucks move on together.

   (i) Find, in ms^{-1}, the speed of the trucks immediately after the collision.

   (ii) Find the impulse exerted on B when the trucks collide.

   (iii) After the collision, the motion of the trucks is opposed by a constant horizontal resistance of magnitude R newtons. If the trucks come to rest 20 s after the collision, find R.

[3, 3, 4 marks]

5. A particle of mass 3 kg moves on a smooth horizontal table under the action of a variable horizontal force whose value at time t seconds is

$$6\cos t\mathbf{i} - 3e^{-t}\mathbf{j} \quad \text{newtons.}$$

When $t = 0$, the particle has a velocity $\mathbf{j}$ ms^{-1} and is at the point with position vector $(3\mathbf{i} - \mathbf{j})$ metres.

(i) Find the velocity and position of the particle at time t.

(ii) Briefly describe the motion of the particle when t is large.

(iii) By integrating the scalar product of the force and the velocity with respect to time, or otherwise, find the work done on the particle in the first second.

[2,3; 2; 3 marks]

6. Two particles A and B, of masses 0.1 kg and 0.2 kg respectively, are connected by a light inextensible string. The particle A is placed near the bottom of a rough plane inclined at 30° to the horizontal. The string passes over a small smooth light pulley which is fixed at the top of the inclined plane, and B hangs freely. The system is released from rest, with each portion of the string taut and in the same vertical plane as a line of greatest slope of the inclined plane. The coefficient of friction between the the particle A and the inclined plane is 1/2. Calculate:

(i) the normal reaction of the inclined plane on A;

(ii) the frictional force on A;

(iii) the common acceleration of the two particles;

(iv) the tension in the string.

[2, 1, 4, 3 marks]

7. A light inextensible string of length $5a$ has one end fixed at a point A, and the other end fixed at a point B which is vertically below A and at a distance $4a$ from it. A particle P of mass m is attached to the midpoint of the string and moves with angular speed ω, in a horizontal circular path whose centre is the midpoint of AB.

(i) Assuming the motion occurs with both parts AP and BP taut, find the tensions in AP and BP in terms of $m, g, a,$ and ω.

(ii) Show that part BP of the string can only remain taut if $\omega^2 > \dfrac{g}{2a}$.

[7, 3 marks]

8. The three points O, B, C lie in that order on a straight line l on a smooth horizontal plane, with OB = 0.3 m and OC = 0.4 m. A particle P describes simple harmonic motion with centre O along the line l. At B the speed of the particle is 12 ms^{-1}, and at C its speed is 9 ms^{-1}. Find:

   (i) the amplitude of the motion;

   (ii) the period of the motion;

   (iii) the maximum speed;

   (iv) the maximum acceleration;

   (v) the time taken to travel from O to C.

[3, 3, 1, 1, 2 marks]

9. A ship A is travelling due east at 24 km/hour. At noon, a second ship B is 8 km away from A in a north-easterly direction, and one hour later, B is again 8 km away from A, but in a south-easterly direction. The speed of B can be assumed to be constant.

   (i) Find the speed of B in km/hour.

   (ii) Calculate the minimum distance between A and B, and show that when this occurs, B is due east of A.

[5, 5 marks]

10. Two uniform rods AB and BC, each of length $2a$, and of mass 2 kg and 3 kg respectively, are smoothly hinged at B. The ends A and C are each smoothly hinged to two points in the same horizontal straight line and at a distance of $2a$ apart. The system is hanging in equilibrium from A and C.

    Find the horizontal and vertical components of the reactions at each hinge.

[10 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL

MAY 2012

SUBJECT:	APPLIED MATHEMATICS
PAPER NUMBER:	II
DATE:	18th May 2012
TIME:	4.00 p.m. to 7.00 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. A *light* beam AB of length $2a$ is clamped horizontally at the end A. The beam carries a concentrated vertical load of weight W at its midpoint, and another vertical load W at the other end B. This cantilever system is in equilibrium.

   (i) Find the external force and couple required at the clamped end of the beam to maintain equilibrium.

   (ii) Find the shearing force and bending moment at any point along the beam.

   (iii) Draw a sketch of the shearing force and bending moment.

   (iv) Find the maximum bending moment and the maximum shearing force, and the point/s on the beam where they occur.

[2, 8, 3, 2 marks]

2. (a) A particle P moves so that its position vector $\mathbf{r}$ at time t is given by

$$\mathbf{r} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t,$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, and ω is a constant. Show that $\mathbf{r} \times \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is independent of t.

(b) A force $\mathbf{F}$ acts at the point with position vector $\mathbf{r}$. Express as a vector product the moment of this force about the point with position vector $\mathbf{a}$.

A force of unit magnitude has equal moments about points with position vectors $(0, 1, 0)$ and $(1, 2, -1)$. Find the possible values of the components of this force.

[7, 8 marks]

3. A fixed smooth solid sphere has radius a and centre O. A particle A of mass m is held at a point P on the surface of sphere, with the line OP making an acute angle $\cos^{-1}\frac{3}{4}$ with the upward vertical. The particle is then released from rest from this position.

(i) Prove that when OA makes an angle θ with the upward vertical, the velocity v of the particle is given by $v^2 = \frac{1}{2}ga(3 - 4\cos\theta)$ provided that the particle remains in contact with the sphere.

(ii) Find the normal reaction while the particle is in contact with the sphere.

(iii) Deduce that the particle leaves the sphere's surface when OA makes an angle of 60° with the upward vertical.

[6, 6, 3 marks]

4. A small bullet is projected from the origin with speed V at an angle of elevation α to the horizontal. The bullet is required to hit a small target at a horizontal distance a, and at a height b above the point of projection. It can be assumed that there is no resistance to the motion of the bullet.

(i) Find from first principles the Cartesian equation of the path followed by the bullet.

(ii) Show that the target cannot be hit if $V^2(V^2 - 2gb) < g^2a^2$, but that if $V^2(V^2 - 2gb) > g^2a^2$, there are two possible angles of elevation.

(iii) Show that if $V^2 = 2ga$ and $b = \frac{3}{4}a$, there is only one possible angle of elevation, and find the time taken to hit the target in this case.

[5, 5, 5 marks]

5. Two smooth uniform spheres A and B, of equal radius, are moving on a smooth horizontal plane. Sphere A has mass 2 kg and velocity $(2\mathbf{i}+\mathbf{j})\text{ms}^{-1}$, and sphere B has mass 3 kg and velocity $(-\mathbf{i}-\mathbf{j})$ ms^{-1}. The spheres collide when the line joining their centres is parallel to $\mathbf{j}$. The coefficient of restitution for this collision is $\frac{1}{2}$.

   (i) Find the velocities of A and B immediately after the collision, giving your answers in vector form.

   (ii) Find the impulse imparted by the collision on sphere B.

[12, 3 marks]

6. A particle A of mass m can move along the x-axis under the influence of a *repulsive* force $\frac{k}{x^2}$, where x is the distance of the particle from a fixed origin O, and k is a positive constant. Initially, the particle is at a distance d from O, and is moving with velocity u towards O.

   (i) Write down the equation of motion for the particle A.

   (ii) By integrating this equation, or by considering conservation of energy, obtain a relation between x and the velocity $\dot{x}$.

   (iii) Deduce the smallest distance of the particle from the origin in the subsequent motion.

[3, 9, 3 marks]

7. A uniform sphere of radius a and weight $W/\sqrt{3}$ rests on a rough horizontal table. A uniform rod AB of weight $2W$ and length $2a$ is freely hinged at A to a fixed point on the table and leans against the sphere so that the centre of the sphere and the rod lie in a vertical plane. The rod makes an angle of 60° with the horizontal.

   (i) Draw a diagram showing clearly the forces acting on the rod and on the sphere.

   (ii) By considering equilibrium of the rod and the sphere separately, show that the normal reaction and the frictional force between the rod and the sphere is $\frac{1}{\sqrt{3}}W$ and $\frac{1}{3}W$ respectively.

   (iii) Find also the normal reaction and the frictional force between the sphere and the table.

   (iv) If the coefficient of friction at each point of contact is μ, find the smallest value of μ which makes equilibrium possible.

[4; 2,2; 2,2; 3 marks]

8. A pendulum consists of a uniform rod of mass m and length $2a$, attached at its end to the circumference of a ring, also of mass m, and radius a. The rod is perpendicular to the circumference of the ring and lies in the same plane as the ring. The pendulum oscillates in the plane of the ring about the free end of the rod.

   (i) Find the period of small oscillations.

   (ii) Find the percentage change in the period if the ring were replaced by a uniform disc of the same size and mass.

[9, 6 marks]

9. A uniform rod AB has mass m and length $2a$. O is a point on the rod such that AO $= \frac{1}{2}a$. A particle of mass m is attached to the end B. The loaded rod is free to rotate about a fixed smooth horizontal axis $\boldsymbol{l}$ passing through O and perpendicular to the rod. Initially, at time $t = 0$, the loaded rod is held at rest in a vertical position with B above O. It is then slightly disturbed. After a time t, the rod makes an angle θ with the upward vertical.

   (a) Show that the moment of inertia of the loaded rod about the axis $\boldsymbol{l}$ is $\frac{17}{6}ma^2$.

   (b) Find the angular velocity, $\dot{\theta}$, of the system at a general time t.

   (c) Find the horizontal component of the reaction on the axis when the rod is horizontal.

[5, 7, 3 marks]

10. A uniform sphere has mass m and radius a.

   (i) Find by integration the moment of inertia of the sphere about a diameter.

   (ii) The sphere is rolling along a straight line on a rough horizontal plane with constant velocity v. It strikes the rough edge of a step, of height $a/5$, which is perpendicular to the direction of the motion. It can be assumed that on and after impact, the sphere does not slip at the point of contact, the friction being sufficient to prevent slipping. It can also be assumed that after impact, the sphere is free to rotate about the edge of the step.

   Using conservation of angular momentum about the edge of the step, find the angular velocity with which the sphere begins to turn about the edge of the step.

[7, 8 marks]