MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL

2018 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	8^{th} May 2018
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Attempt **ALL** questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10\ \text{ms}^{-2}$)

1. A uniform solid cylinder has mass $6m$, base radius r and height $2r$, whilst a uniform solid cone has mass $4m$, base radius r and height r. The flat face of the cone is then attached to a plane face of the cylinder, in such a way that the circular rims of the cylinder and the cone coincide.

   (a) Find the centre of gravity of the resulting solid. (7)

   (b) The solid is now placed with its plane face in contact with a horizontal table. The surface of the table is rough enough to prevent the body slipping as the table is slowly tilted. Find the angle through which the table has been tilted when the body is on the point of toppling.

   (3)

(Total: 10 marks)

2. A Cartesian system of coordinates has origin O and has unit vectors **i** and **j** along the x-axis and y-axis respectively. Forces act in the x-y plane as follows:

   3**i**+3**j** N at the point with position vector 3**i**+**j** m,

   4**i**–5**j** N at the point with position vector **i**+3**j** m,

   –5**i**+2**j** N at the point with position vector –2**i**+**j** m, and

   2**i**+3**j** N at the point with position vector –2**i**–2**j** m.

   (a) Find the single force that could replace this system and find where its line of action cuts the x-axis. You can optionally use the vector product to find the moment of these forces about O. (5)

   (b) A couple of moment a Nm anticlockwise and a force b**i**+c**j** N acting at the point with position vector 2**i**+**j** m, are now added to the system. If these reduce the system to equilibrium, find a, b and c. (5)

   (Total: 10 marks)

3. O and B are two points on level ground, with OB $= 4a$. A vertical tower of height $4h$ has its base at O, and a vertical tower of height h has its base at B. A Cartesian coordinate system is taken with origin at O, the x-axis along OB, and the y-axis vertically upwards. At time t=0, two particles are projected horizontally in opposite directions from the top of the towers. The initial speed of the particle projected from the higher tower is u_1, whilst that of the particle projected from the lower tower is u_2. Both particles hit the ground at the point X between O and B, with OX $= 3a$.

   (a) Find the velocity and displacement of the two particles in terms of a, u_1, u_2, g and t. (4)

   (b) Find the times of flight of the two particles in terms of these quantities. (3)

   (c) Show that $3u_2 = 2u_1$. (3)

   (Total: 10 marks)

4. A uniform ladder AB is of weight $2W$ N and length 10 m. It rests with end A on a rough horizontal floor, and end B against a rough vertical wall. The coefficient of friction at the wall and at the floor is 1/3, and the ladder makes an angle θ with the horizontal, where $\tan\theta = 16/7$. A man of $5W$ N starts to climb up the ladder.

   How far up the ladder can the man climb before before slipping occurs? You can assume that friction is limiting at both ends of the ladder. **(Total: 10 marks)**

5. A car of mass 800 kg is pulling a trailer of mass 200 kg up a hill inclined at an angle $\sin^{-1}(1/14)$ to the horizontal. When the total force exerted by the engine is 1000 N, the car and trailer move up the hill at a steady speed of 10 ms^{-1}.

   (a) Find the total frictional resistance to the motion of the car and trailer during this motion. (3)

   (b) Calculate the power exerted by the engine. (2)

   (c) The car and trailer are travelling at 10 ms^{-1} when the power is instantaneously increased to 12 kW. Find the instantaneous acceleration. (3)

   (d) Find the instantaneous tension in the coupling between the car and the trailer, if the total frictional force on the trailer is 75 N. (2)

(Total: 10 marks)

6. A light inextensible string AB of length 33 cm has a particle of mass 50 g attached to it at a point P, 13 cm from the end A. The ends of the string are attached at two fixed points in the same vertical line, with A 21 cm above B. The particle moves in a horizontal circle with both parts of the string taut at a constant speed of $2\sqrt{3}$ ms^{-1}.

   (a) Using the cosine rule or otherwise, find $\angle$ PAB, and the radius of the circle. (3)

   (b) Find the tensions in the two parts of the string. (7)

(Total: 10 marks)

7. A light framework consists of five light inextensible rods, smoothly hinged together, four of which form the edges of a parallelogram ABCD, the fifth rod being the diagonal BD of the parallelogram.

   Rods AB and DC are horizontal, AD and BC are inclined at 60° to the horizontal, and the diagonal BD is vertical with D above B. The framework is hinged at A to a smooth vertical wall, and rests on a smooth support at B. The system carries a vertical load of 3 kN at C, and is in equilibrium.

   Find the reactions at A and B, and the forces in the rods, indicating whether they are in tension or compression.

(Total: 10 marks)

8. A and B are two fixed points 10 m apart with A vertically above B. A particle, P, of mass m kg rests in equilibrium between A and B, held in position by two vertical, light, elastic strings, AP and PB. The upper string, AP, has modulus $6mg$ N and natural length 3 m. The lower string, BP, has modulus $12mg$ and natural length 4 m.

   (a) Show that the distance AP at equilibrium is 5 m. (5)

   (b) The body is pulled down a *small* distance of x m and released from rest. Show that the subsequent motion is simple harmonic, and find the periodic time of these oscillations. (5)

(Total: 10 marks)

9. A man swimming with constant speed u relative to still water crosses a straight river of constant width c having parallel banks. The river has a constant current of ku. He would like to swim along the shortest possible path.

   (a) If $k = 1/2$, show that the time taken to reach the opposite bank is $2c/(u\sqrt{3})$. (5)

   (b) If $k = 2$, find the direction in which he should swim, and show that the time taken to cross the river is the same as in part (a). (5)

(Total: 10 marks)

10. A and C are two points in a vertical line, with C above A and AC $= 4a$. A smooth uniform rod AB, of length $3a$ and weight W, is pivoted at A so that it can rotate in a vertical plane. A light ring is free to slide along the rod. A light inextensible string has one end attached to the ring and passes over a smooth peg at C, the other end of the string carrying a particle of weight w which hangs freely.

    (a) By considering the equilibrium of the ring, show that the string is at right angles to the rod. (4)

    (b) By taking moments about A, show that the rod makes an angle θ with the vertical where $\tan\theta = 8w/3W$. (3)

    (c) Find the smallest possible value of the ratio w/W for which equilibrium is possible. (3)

(Total: 10 marks)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2018 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	12th May 2018
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

In this paper, **i**, **j**, **k** are unit vectors along the x-, y- and z-axes of a Cartesian coordinate system.

(Take $g = 10\ \text{ms}^{-2}$)

1. A uniform horizontal beam ABC is of length $4a$ and weight $4W$. It is supported in a horizontal position by vertical supports at B and C, where AB $= a$. A particle of weight $2W$ is suspended from the midpoint of the beam.

   (a) Find the normal reactions at the supports B and C. (2)

   (b) Find the shearing force and bending moment at any point along the beam. (9)

   (c) Draw a sketch of the shearing force and bending moment, and deduce where the bending moment is largest. (4)

(Total: 15 marks)

2. The points P_1 and P_2 have position vectors $2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $-8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ N, whilst the points Q_1 and Q_2 have position vectors $-13\mathbf{j} + 5\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ respectively.

   (a) Show that the lines P_1P_2 and Q_1Q_2 have perpendicular directions. (3)

   (b) Show that the two lines intersect, and find the position vector of their point of intersection. (8)

   (c) A force of magnitude F acting in the direction P_1Q_2 moves a particle along the straight line from P_2 to Q_2. Find the work done by the force. (4)

(Total: 15 marks)

3. The vertices of a tetrahedron $ABCD$ have position vectors **a**, **b**, **c**, **d** respectively where:

$$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = 4\mathbf{i} + \mathbf{k} \quad \text{and} \quad \mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

   Forces of magnitude 30 and $3\sqrt{13}$ act along $\overrightarrow{CB}$ and $\overrightarrow{CD}$ respectively. A third force acts at A.

   (a) If the system reduces to a couple, find the magnitude of this couple and the force at A. (12)

   (b) Find the equation of the plane which passes through C and which has its normal parallel to the axis of the couple. (3)

(Total: 15 marks)

4. A pendulum consists of a light inextensible *rod* OA, of length 2 m, with a particle of mass 3 kg attached at A. The system is freely pivoted at O and can rotate freely in a vertical plane. When A is at its lowest point, it has a speed of u ms^{-1}. Subsequently, when OA makes an angle of θ with the downward vertical, the speed of A is v ms^{-1}.

   (a) Using conservation of energy, obtain a relation for v in terms of u, g and θ. Find the smallest value of u for the pendulum to perform complete revolutions about O. (7)

   (b) If the rod is now replaced by a light inextensible *string* of the same length, obtain, using Newton's Second Law, an expression for the tension in the string in terms of u, g and θ. Deduce the smallest value of u for the pendulum to perform complete revolutions. (8)

(Total: 15 marks)

5. Two spheres A and B, of masses 3 gm and 2 gm respectively, but of equal radius, are moving along a smooth horizontal plane. The x- and y-axes of a Cartesian coordinate system, with corresponding unit vectors $\mathbf{i}$ and $\mathbf{j}$, lie on this plane.

   When A has a velocity $3\mathbf{i} - \mathbf{j}$ cm/sec, it collides with sphere B which is moving with velocity $2\mathbf{i} + \mathbf{j}$ cm/sec. At the instant of collision the line of centres of the two spheres is in the direction $\mathbf{i} + \mathbf{j}$, and the coefficient of restitution between them is $1/2$.

   (a) Find the velocity vectors of the two spheres after the collision. (12)

   (b) Assuming the spheres are very small, find the distance between the two spheres one second after the collision. (3)

(Total: 15 marks)

6. A ball of mass m is thrown vertically upwards. When the ball is at a height x above its initial position, its speed is v. The only forces acting on the ball are its weight and a resisting force of magnitude mkv, where k is a positive constant.

   (a) Using Newton's Second Law of Motion for the upward motion of the ball, set up a differential equation relating the speed v to the displacement x of the ball. (4)

   (b) Integrate this differential equation to find v in terms of x, given that the initial speed of the ball is $3g/k$. (9)

   (c) Show that the maximum height of the ball above its initial position is (2)

$$\frac{g}{k^2}(3 - \ln 4).$$

(Total: 15 marks)

7. A uniform rod has centre G, mass M and length $2a$. P is a point on the rod with GP $= x$.

   (a) Using integration, find the moment of inertia of the rod about an axis I_G passing through G and perpendicular to the rod. Hence find by the parallel axes theorem, the moment of inertia about a parallel axis I_P passing through P. (5)

   (b) The rod can rotate freely in a vertical plane about a smooth fixed horizontal axis I_P passing through P. It performs small oscillations about its equilibrium position. Find the period of these oscillations. (6)

   (c) Show that the period is least when $x = a/\sqrt{3}$. (4)

(Total: 15 marks)

8. (a) A uniform annular ring has mass per unit area ρ, internal radius a and external radius $2a$. Find by integration the moment of inertia of the ring about an axis perpendicular to it and passing through its centre. Express your answer in terms of ρ and a. If the mass of the ring is given to be $6M$, find by expressing ρ in terms of M, the moment of inertia of the ring in terms of M and a. (5)

   (b) An automobile gear system consists of the ring described in part (a), together with a uniform disc of mass M and radius a. Initially, the disc is spinning in a horizontal plane with angular velocity ω_0 about a fixed smooth vertical axis passing through its centre. Initially, the ring is at rest horizontally, with its centre vertically over the centre of the rotating disc, and is then dropped. When the annulus circumscribes the disc, there is no slipping at the circle of contact. Find the angular velocity of the compound disc. (6)

   (c) Find the kinetic energy lost by the system. (4)

(Total: 15 marks)

9. A uniform rod AB, of mass m and length $2a$, is freely pivoted to a fixed point at A and is initially hanging in equilibrium. A particle of mass m moving horizontally with speed u, strikes the rod at its middle point and rebounds from it. The impact is perfectly elastic, with the coefficient of restitution equal to 1.

   (a) Show that the speed of the particle immediately after impact is $u/7$. (5)

   (b) Find the angular velocity with which the rod starts to rotate. (5)

   (c) If the rod comes to instantaneous rest when it is horizontal, show that

$$24u^2 = 49ag. \qquad (5)$$

(Total: 15 marks)

10. A uniform ladder rests with its top end against a rough vertical wall with coefficient of friction μ_1, and its base on a rough horizontal floor with coefficient of friction μ_2. The ladder makes an angle θ with the floor.

    If the ladder is about to slip at both ends, show that $\tan\theta = \dfrac{1-\mu_1\mu_2}{2\mu_2}$.

(Total: 15 marks)