MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL

2019 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	2^{nd} September 2019
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Attempt **ALL** questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

Unless otherwise stated, **i** and **j** are unit vectors along the x- and y- axes of a Cartesian coordinate system. Units are Newtons for force and meters for distance.

(Take $g = 10$ ms^{-2})

1. In a Cartesian system of coordinates with origin O, point A has coordinates $(12, 5)$, whilst point B has coordinates $(6, 8)$. Three forces act on the system as follows:

$$6.5 \text{ N along } \overrightarrow{\text{OA}}, \qquad 10 \text{ N along } \overrightarrow{\text{OB}}, \qquad 4\sqrt{5} \text{ N along } \overrightarrow{\text{AB}}.$$

The forces are given in Newtons, and distances in meters.

(a) Express these forces and their resultant in **i** and **j** notation. (5)

(b) Find the equation of the line of action of the resultant. (5)

(Total: 10 marks)

2. A Cartesian system of coordinates has origin O, and has the x-axis horizontal and the y-axis vertical.

   (a) A particle P is projected from O with speed V at an angle θ to the horizontal. Obtain from first principles, the velocity and displacement of the particle in **i**, **j** notation, in terms of the time t after projection. (4)

   (b) A second particle Q is projected from O simultaneously with particle P, with the same speed V, but at an angle α to the horizontal, where $\alpha \neq \theta$. Show that:

   (i) the line joining the two particles has a constant gradient; (3)

   (ii) the distance between them increases at a constant rate. (3)

(Total: 10 marks)

3. A string is threaded through a smooth ring of weight W, and is tied to two points A and B on the same level. The ring is pulled by a horizontal force P, so that the two parts of the string are inclined at angles of 60° and 30° respectively to the vertical. The tension in the string is T. Find T and P if:

   (a) A and B are on opposite sides of the vertical through the ring; (6)

   (b) A and B are on the same side of the vertical through the ring. (4)

(Total: 10 marks)

4. A tow truck of mass 1600 kg is pulling a damaged car of mass 800 kg along a level road. The driving force of the tow truck is 7000 N, whilst the resistances to motion on the tow truck and the car are 1000 N and 1200 N respectively.

   (a) Show that the common acceleration of the tow truck and the car is 2 ms^{-2}. (1)

   (b) The tow truck and the car are connected by light horizontal coupling. Find the tension in the coupling. (1)

   (c) The coupling breaks at the instant when the speed is 2.7 ms^{-1}. Assuming that the resistances to motion and the driving force do **not** change, find the acceleration of the tow truck and the deceleration of the damaged car when the coupling breaks.

   (4)

   (d) Calculate the distance between the tow truck and the car when the latter comes to rest. (4)

(Total: 10 marks)

5. Particles A and B have masses of 3 kg and 2 kg respectively, and lie at rest on a smooth horizontal plane. The particles are connected together by a light inextensible string, which is initially slack. A is projected towards B with a velocity of 10 ms^{-1}. The coefficient of restitution of the particles is 0.5. Using conservation of momentum, find:

   (a) the velocities of the particles immediately after impact; (5)

   (b) the common velocity of the particles after the string becomes taut; (3)

   (c) the impulse in the string when it becomes taut. (2)

(Total: 10 marks)

6. Four particles, each of mass 2 kg, are connected by light inextensible strings, each of length 0.10 m, so that they form a square with the particles at the corners, and the strings forming the sides of the square. The particles are placed in this configuration symmetrically on a smooth turntable, which is made to rotate with an angular speed of 5 rad s^{-1}.

   Find the tension in the strings.

(Total: 10 marks)

7. Two rods AB and AC are smoothly jointed together at A.

   The rod AB has length a and weight W, whilst AC has length $2a$ and weight $2W$.

   The system hangs from two fixed smooth hinges at B and C, which are on the same horizontal level, and are positioned so that angle BAC is 90°.

   Find the horizontal and vertical components of the reactions at A, B and C.

(Total: 10 marks)

8. A wooden block of mass 40 kg is pulled along a straight horizontal road by a rope. The coefficient of friction between the block and the road is 0.1, and the acceleration of the block is 0.12 ms^{-2}. Find the tension in the rope:

   (a) if the rope is horizontal; (5)

   (b) if the rope makes an angle of 20° with the horizontal. (5)

(Total: 10 marks)

9. A particle P of mass 4 kg is attached to one end of a light elastic string OP, the other end being attached to a fixed point O on a smooth plane inclined at 30° to the horizontal. The particle can move along a line of greatest slope of the inclined plane.

   The modulus of elasticity of the string is 40 N, and its natural length is 2 m.

   (a) If the system is in equilibrium, find the extension in the string. (4)

   (b) The particle is then displaced slightly, so that it performs small oscillations along the line of greatest slope. Find the periodic time of these oscillations. (6)

(Total: 10 marks)

10. A uniform lamina has mass M, and mass per unit area ρ. The lamina lies in the x-y plane, and is bounded by the x-axis, the line $x = 1$ and the curve $y = x^2$. Using integration:

    (a) show that $\rho = 3M$; (3)

    (b) find the coordinates of the centroid of the lamina. (7)

(Total: 10 marks)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL

2019 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	3rd September 2019
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

In this paper, **i**, **j** and **k** are unit vectors along the x-, y- and z-axes of a Cartesian coordinate system.

(Take $g = 10\ \text{ms}^{-2}$)

1. A uniform heavy springboard AB of length $2l$ and weight $2W$ is clamped horizontally at A. A diver of weight W stands at the midpoint of the springboard. This cantilever system is in equilibrium.

   (a) Find the force and couple exerted by the clamp. (2)

   (b) Find expressions for the shearing force and bending moment at any point along the springboard. (9)

   (c) Draw a sketch of the shearing force and the bending moment along the springboard. (4)

(Total: 15 marks)

2. A system consists of two forces as follows:

$$\mathbf{F}_1 = 5\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} \quad \text{acting at} \quad \mathbf{r}_1 = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k},$$

$$\mathbf{F}_2 = 4\mathbf{i} + \mathbf{j} - \mathbf{k} \quad \text{acting at} \quad \mathbf{r}_2 = 11\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}.$$

(a) Show that the lines of action of these forces are coplanar. (9)

(b) Find the resultant of this system and the vector equation of its line of action. (3)

(c) Find the moment of the resultant about the origin. (3)

(Total: 15 marks)

3. At time t, two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where

$$\mathbf{a} = 2a\mathbf{i} + a\cos\omega t\ \mathbf{j} + a\sin\omega t\ \mathbf{k},$$

$$\mathbf{b} = a\sin\omega t\ \mathbf{i} - a\cos\omega t\ \mathbf{j} + 3a\ \mathbf{k},$$

and a and ω are constants.

(a) Find $\mathbf{r}$, the position vector of A relative to B, and the distance between A and B. (4)

(b) Find $\mathbf{v}$, the velocity of A relative to B. (3)

(c) Find the values of t for which $\mathbf{r}$ and $\mathbf{v}$ are perpendicular. (4)

(d) Deduce the smallest and greatest distances between A and B, and the times at which they occur. (4)

(Total: 15 marks)

4. A smooth sphere of mass 3 kg moving on a horizontal plane with velocity $3\mathbf{i} + \mathbf{j}$ ms^{-1} collides with another smooth sphere of mass 2 kg moving with velocity $2\mathbf{i} + \mathbf{j}$ ms^{-1}. At the instant of collision, the line of centres of the two spheres is in the direction $\mathbf{i} + \mathbf{j}$, and the coefficient of restitution is 0.5.

(a) Find unit vectors parallel and perpendicular to the line of centres. (2)

(b) Using the scalar product, or otherwise, find the components of the velocities before impact in the directions parallel and perpendicular to the line of centres. (4)

(c) Find the velocities of the spheres after impact. (7)

(d) Find the kinetic energy lost as a result of the impact. (2)

(Total: 15 marks)

5. A parachutist of mass m opens her parachute when she is falling with a velocity $4V$, where V is the terminal velocity with the parachute open. The air resistance R can be assumed to be of the form $R = -kv$, where k is a positive constant, and v is the velocity of the parachutist.

   (a) Using Newton's Second Law, set up a differential equation for the motion of this parachutist. (2)

   (b) Express k in terms of m, g and V. (2)

   (c) By integrating once with respect to t, using separation of variables, obtain a relation for v in terms of t, g and V. (9)

   (d) Show that your solution satisfies the initial conditions. (2)

(Total: 15 marks)

6. Two uniform spheres of radius a and weight W rest on rough horizontal ground with their centres distant $2\sqrt{2}a$ apart. A third sphere of radius a and weight W is balanced on top of the other spheres such that all three spheres lie in the same vertical plane.

   One of the lower spheres, B, has centre O. It touches the ground at the point P, and touches the upper sphere at the point Q.

   (a) Assuming that friction is **not** limiting at any contact, draw a diagram showing carefully the reactions and forces acting on each sphere. (6)

   (b) Consider equilbrium for sphere B. By taking moments about its centre and by resolving forces horizontally, find the ratio between the normal reaction and the frictional component at Q. (4)

   (c) By taking moments about Q for sphere B, and resolving vertically for the whole system, find the ratio between the normal reaction and the frictional component at P. (4)

   (d) Using (b) and (c), deduce the minimum value of the coefficient of friction, μ, if equilibrium is to be maintained. (1)

(Total: 15 marks)

7. A light rod AB can turn freely in a vertical plane about a smooth hinge at A and carries a mass m hanging from B. A light string of length $2a$ fastened to the rod at B passes over a smooth peg at a point C vertically above A and carries a mass km at its free end. Side $AB = AC = a$, and $\angle BAC = 2\theta$.

   (a) Obtain an expression for the potential energy of the system, $U(\theta)$, in terms of θ, k, m, a and g. (8)

   (b) By finding $\dfrac{dU}{d\theta}$ and equating it to zero, find the positions of equilibrium of the system. Here you can use the identity $\sin 2\theta = 2\sin\theta\cos\theta$. (3)

   (c) If equilibrium is possible with the rod horizontal, find the value of k. (2)

   (d) Find the second derivative $\dfrac{d^2U}{d\theta^2}$. Show that it is negative for the case when the rod is horizontal. This implies that equilibrium is unstable for this position. (2)

(Total: 15 marks)

8. $ABCD$ is a uniform lamina, with mass M and mass per unit area ρ. In this lamina,

$$AB = AD = 2a, \quad CB = CD = a, \quad \angle ABC = \angle ADC = 90^\circ.$$

   In the following it can be assumed that the moment of inertia of a uniform triangle PQR about PQ is equal to $\frac{1}{6}Mh^2$, where M is its mass, and h is the perpendicular distance from R to PQ.

   (a) Obtain a relation between M and ρ. Hence find the masses of ABD and CBD in terms of M. (4)

   (b) Show that the moment of inertia of $ABCD$ about BD is $\frac{13}{30}Ma^2$. (6)

   (c) Find the moment of inertia of $ABCD$ about AC. (3)

   (d) Find the moment of inertia of $ABCD$ about an axis perpendicular to the lamina and passing through the point of intersection of AC and BD. (2)

(Total: 15 marks)

9. One end of a light inextensible string is fastened to a point on the rim of a uniform circular disc of mass m, radius a and centre O. The string is wrapped several times round the circumference and the other end is then attached to a fixed point A so that the portion of string which is not in contact with the disc is taut and vertical. The disc is then released from rest.

   (a) Assuming that no slipping occurs between the string and the disc, obtain a relation between $\dot{\theta}$, the angular velocity of the disc, and $\dot{x}$, the downward velocity of O. (3)

   (b) Using Newton's Second Law, obtain an equation relating the tension in the string to the acceleration of O. (5)

   (c) By taking moments about O, obtain an equation relating the tension in the string to the angular acceleration of the disc. (5)

   (d) Find the tension in the string and the linear acceleration of O. (2)

(Total: 15 marks)

10. A uniform solid sphere of mass m and radius a has moment of inertia $\frac{2}{5}ma^2$ about a diameter.

    The sphere is rotating freely about a fixed horizontal diameter with angular velocity ω_0. A stationary particle of mass m adheres to the lowest point of the sphere.

    (a) Find the angular speed of the sphere immediately after picking up the particle. (6)

    (b) Find, in terms of a, ω_0 and g, the cosine of the angle that the radius to the particle makes with the downward vertical when the sphere comes to instantaneous rest.

    (6)

    (c) Show that the sphere will make complete revolutions if $\omega_0^2 > \dfrac{35g}{a}$. (3)

(Total: 15 marks)