MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	4th October 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y- and z-axis of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$)

1. In the rectangle $ABCD$, $AB = 2a$ and $AD = a$. Forces K, L, M, N act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, respectively.
   (a) Find the equations which must be satisfied by K, L, M, N, if:
      (i) the forces reduce to a couple;
      [1 mark]
      (ii) the forces are in equilibrium;
      [3 marks]
      (iii) the forces reduce to a resultant force acting through the mid-points of AB and CD.
      [3 marks]
   (b) If the forces reduce to a resultant force $2K$ acting along $\overrightarrow{AB}$, express L, M and N in terms of K.
      [3 marks]

2. (a) Three light rigid rods form a smoothly-jointed framework ABC in which $AB = 5\,\text{m}$, $AC = 10\,\text{m}$ and $\angle ABC = 90°$. Weights of $100\,\text{N}$ and $50\,\text{N}$ are attached at B and$C$, respectively. Find where the resultant of the two weights cuts BC.

[2 marks]

(b) The framework is freely suspended from A and is in equilibrium in a vertical plane. Show that AB makes an angle of $30°$ with the vertical. Find the magnitude of the force supporting the framework at A and the forces in the rods, stating which are in tension and which are in compression.

[8 marks]

3. A uniform lamina $ABCED$ of weight W has the shape of a square $ABCD$, of side $2d$, on which a triangle CED has been described externally to the side CD of the square. It is given that $CE = ED$ and $\angle CED = 2\alpha$. The lamina is placed in a vertical plane with the side AD in contact with a horizontal table.

(a) Find the centroid of the lamina from CD in terms of d and α.

[4 marks]

(b) The lamina topples in that plane about D. Prove that $\tan\alpha < \sqrt{3}/6$.

[3 marks]

(c) To prevent this toppling it is required to apply the least possible force to the lamina at one of the points A, B, C or E. Find which point should be chosen and calculate the minimum force required in terms of W and α.

[3 marks]

4. A particle moving in a horizontal straight line, with uniform acceleration f, passes points A, B, C and D which are at distances a, b, c and d respectively from its initial position. The particle takes n seconds to cover each of the distances AB, BC and CD.

(a) Given that the velocity of the particle at A is u, find its velocity at B, C and D in terms of n, u and f.

[2 marks]

(b) Show that $d - a = 3(c - b)$.

[4 marks]

(c) If the velocity of the particle, when at D, is three times the velocity at A, show that $3(b - a) = 2(c - b)$.

[4 marks]

5. (a) A uniform rod AB of length $2l$ and weight W rests in limiting equilibrium at an angle α to the horizontal with one end A on a rough horizontal floor and the other end B against a smooth vertical wall. The rod is in a vertical plane perpendicular to the wall. Find the coefficient of friction between the rod and the floor in terms of α.

[4 marks]

(b) When the rod is in this position, an upward vertical force of magnitude $W/4$ is applied to it at a point C, where $AC = 3l/2$. Show that the equilibrium is no longer limiting, but that if the direction of this force is reversed the rod will slip.

[6 marks]

6. Two particles, A of mass $2m$ and B of mass m, move on a smooth horizontal table in opposite directions with speeds $5u$ and $3u$ respectively. The particles collide directly.

(a) Find their velocities after the collision in terms of u and e, where e is the coefficient of restitution between the particles.

[4 marks]

(b) Find the magnitude of the impulse exerted by B on A.

[2 marks]

(c) Find the value of e if the speed of B after the collision is $3u$.

[1 mark]

(d) Whilst moving at this speed, B collides and coalesces with a stationary particle C of mass $5m$. Find their common velocity.

[3 marks]

7. A boy throws a ball with initial speed $2\sqrt{ag}$ at an angle θ to the horizontal. It strikes a smooth vertical wall and returns to his hand.

(a) By considering the vertical motion show that the total time of flight is $4\sqrt{a/g}\,\sin\theta$.

[3 marks]

(b) By considering the horizontal motion, show that if the boy is standing at a distance a from the wall, the coefficient of restitution between the ball and the wall equals $1/(4\sin 2\theta - 1)$.

[5 marks]

(c) Deduce that the angle θ **cannot** be less than $15°$.

[2 marks]

8. A particle of mass 3 kg moves on a smooth horizontal table under the action of a variable horizontal force given by $(6\cos t\,\mathbf{i} - 3e^{-t}\mathbf{j})$ N. When $t = 0$, the particle has a velocity $\mathbf{j}$ m s^{-1} and is at the point with position vector $(3\mathbf{i} - \mathbf{j})$ m.
   (a) Find the velocity and position of the particle at time t.
   [5 marks]
   (b) Briefly describe the motion of the particle when t is large.
   [2 marks]
   (c) By integrating the scalar product of the force and the velocity with respect to time, or otherwise, find the work done on the particle during the first second.
   [3 marks]

9. A car of weight W has a maximum power H. In all circumstances, there is a constant resistance of magnitude R.
   (a) When the car is moving up a slope inclined at an angle α to the horizontal with $\sin\alpha = \frac{1}{10}$, its maximum speed is v and when it is moving down the same slope, its maximum speed is $2v$. Find R in terms of W.
   [6 marks]
   (b) The maximum speed of the car on level ground is u. Find the maximum acceleration of the car when it is moving with speed $\frac{1}{2}u$ up the given slope.
   [4 marks]

10. A light elastic string has natural length a and modulus λ. One end of this string is fastened to a fixed point A, and a mass m is attached to the other end. The particle is released from rest at A, and first comes to rest when it has fallen a distance $3a$.
   (a) Using conservation of energy, or otherwise, show that $\lambda = \frac{3}{2}mg$.
   [3 marks]
   (b) Show that at the lowest point of its path, the acceleration of the particle is $2g$ upwards.
   [3 marks]
   (c) Find in terms of g and a the speed of the particle when the magnitude of the acceleration is $\dfrac{g}{2}$.
   [4 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	5^{th} October 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions and each question carries 15 marks.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x-, y- and z-axis of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$)

1. Forces $\mathbf{F}_1 = (2\mathbf{i} + \mathbf{j} + \mathbf{k})\,\mathrm{N}$ and $\mathbf{F}_2 = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\,\mathrm{N}$ act at the points with position vectors $\mathbf{r}_1 = (4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k})\,\mathrm{m}$ and $\mathbf{r}_2 = (-3\mathbf{i} + c\mathbf{j})\,\mathrm{m}$, respectively.
   (a) Show that, if the lines of action of the forces intersect, then $c = -4$.
   [5 marks]
   (b) Find the magnitude of the resultant and the vector equation of its line of action.
   [4 marks]
   (c) Find the vector moment of the resultant about the origin and also about the point with position vector $(-\mathbf{i} - 10\mathbf{j} + 6\mathbf{k})\,\mathrm{m}$.
   [6 marks]

2. A light beam $ABCD$ of length 20 m rests horizontally on supports at A and C, where $AB = 10$ m and $BC = 5$ m. The beam carries a uniform distributed load of $10\,\mathrm{N\,m^{-1}}$ between B and D and concentrated loads of 30 N and 60 N at A and B, respectively.
   (a) Show that the reactions at A and C are 50 N and 140 N, respectively.
   [2 marks]
   (b) Draw the shearing force and bending moment diagrams.
   [10 marks]
   (c) Find the magnitudes of the maximum and minimum bending moments and the points where these values occur.
   [3 marks]

3. A hemispherical bowl of radius r is fixed with its circular rim uppermost, and lying in a horizontal plane. A uniform rod of length l and weight W rests over the rim of the bowl, and has one end in contact with the inner surface of the bowl. All contacts are smooth and the rod is inclined at an angle θ to the horizontal.
   (a) Draw a diagram showing carefully the forces acting on the system.
   [5 marks]
   (b) Show that $4r\cos 2\theta = l\cos\theta$.
   [5 marks]
   (c) Using Lami's Theorem, or otherwise, find the reactions on the rod in terms of W and θ.
   [5 marks]

4. At time $t = 0$, the position vectors of two particles A and B relative to an origin O are $(3\mathbf{i} - 7\mathbf{j} - 4\mathbf{k})$ km and $(-\mathbf{i} + \mathbf{j} + p\mathbf{k})$ km respectively. The particles are moving with constant velocities $(\mathbf{i}+\mathbf{j}+\mathbf{k})\,\mathrm{km\,h^{-1}}$ and $(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\,\mathrm{km\,h^{-1}}$ respectively.
   (a) If the particles collide at some time $t > 0$, find the value of p, and find the coordinates of the point where they collide.
   [6 marks]
   (b) If $p = 1$, find the shortest distance between A and B in the subsequent motion giving your answer correct to two decimal places.
   [9 marks]

5. A uniform rod AB of mass m and length $2l$ is smoothly jointed at A to a fixed point. A light elastic string of natural length l and modulus of elasticity mg connects the end B to a point C distant $2l$ vertically above A. The angle $B\hat{A}C$ is denoted by 2θ, whilst the total potential energy (elastic plus gravitational potential energy) is denoted by V.
   (a) Obtain an expression for V in terms of m, g, l and θ.
   [8 marks]
   (b) Find $\dfrac{\mathrm{d}V}{\mathrm{d}\theta}$.
   [2 marks]
   (c) By setting this derivative equal to zero, verify that $\theta = \frac{\pi}{2}$ and $\theta = \sin^{-1}(\frac{1}{3})$ are positions of equilibrium.
   [2 marks]
   (d) Find $\dfrac{\mathrm{d}^2V}{\mathrm{d}\theta^2}$ and hence deduce for which of the angles mentioned in part (c) the equilibrium of the system is stable.
   [3 marks]

6. Two identical smooth spheres A and B each of mass m lie on a horizontal table. Sphere A is moving with velocity $2u(\cos\alpha\mathbf{i} - \sin\alpha\mathbf{j})$ and sphere B with velocity $u(-\cos\alpha\mathbf{i} - \sin\alpha\mathbf{j})$. They collide when the line joining their centres is in the direction parallel to $\mathbf{i}$.
   (a) If, after impact, sphere A is moving in a direction perpendicular to $\mathbf{i}$, show that the coefficient of restitution is $\frac{1}{3}$ and that the kinetic energy of sphere B is unchanged by the collision.

   [7 marks]

   (b) Find the total loss in the kinetic energy of the spheres in the collision and show that this is maximum when the two spheres collide directly.

   [8 marks]

7. (a) Prove that the work done by a force acting on a body of constant mass is equal to the change in the kinetic energy of the body.

   [3 marks]

   (b) An engine working at constant power H draws a train of total mass M against a constant resistance R. Show that, if the train has speed v when it has travelled a distance x from rest in time t, its kinetic energy is $Ht - Rx$.

   [3 marks]

   (c) By differentiating the kinetic energy with respect to t, show that v and t are related by the differential equation

$$Mv\frac{\mathrm{d}v}{\mathrm{d}t} = H - Rv\,.$$

   [3 marks]

   (d) By solving the differential equation, show that the train reaches a speed V in time

$$\frac{MH}{R^2}\ln\left(\frac{H}{H-RV}\right) - \frac{MV}{R}\,.$$

   [6 marks]

8. A light elastic spring, of modulus $8mg$ and natural length l, has one end attached to a ceiling and carries a scale pan of mass m at the other end. The scale is slightly pulled vertically downwards and then released. In the subsequent motion the pan performs vertical oscillations with period T.

   (a) Show that $T = 2\pi\sqrt{\dfrac{l}{8g}}$.

   [6 marks]

   (b) A weight of mass km is placed in the scale pan, and the above procedure is repeated from the new equilibrium position. The period of oscillation is now $2T$. Find the value of k.

   [4 marks]

   (c) Find the maximum amplitude of the latter oscillations if the weight is in contact with the scale pan throughout the subsequent motion.

   [5 marks]

9. A uniform rod AB, of length $2a$ and mass m, is free to rotate about a fixed axis horizontal through A. The rod is held horizontal and then set in motion with initial angular velocity ω, so that B starts moving downwards. When the rod becomes vertical, B collides with a stationary particle of mass m, which adheres to it. In the subsequent motion the rod is next instantaneously at rest in the horizontal position. It can be assumed that throughout the motion the rod is always normal to the axis of rotation.

   Obtain an expression for ω in terms of a and g.

   [15 marks]

10. (a) Find from first principles the moment of inertia of a uniform circular disc of mass m and radius a about an axis perpendicular to the disc and passing through its centre.

   [4 marks]

   (b) A mass $2m$ rests on a smooth horizontal table. It is connected by a light inelastic string which passes over a rough pulley at the edge of the table to a mass m hanging freely. The pulley is a uniform circular disc of mass m and radius a. It can be assumed that slipping does **not** occur between the string and the pulley.

   Find the acceleration of the mass m, and find the tension in the vertical portion of the string.

   [11 marks]