MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2022 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	30^{th} April 2022
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions is 100.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A mass of 30 kg lies on the rough surface of a plane inclined at 30° to the horizontal. The mass is prevented from sliding down the plane by a horizontal force F acting on it. The coefficient of friction between the mass and the plane is 0.3. Find

   (a) the minimum value of the horizontal force F; (5)

   (b) the normal reaction between the particle and the plane for this value of F. (5)

   (**Total: 10 marks**)

2. An L-shaped beam ABC of uniform density has an angle of 90° at B. The sections AB and BC, both have a length of 1 m and a mass of 50 kg each.

   (a) The beam is freely and smoothly suspended at point A. Find the angle that AB makes with the vertical. (5)

   (b) The point C is then pulled by a force F until BC is horizontal, while ABC remains in a vertical plane. The angle that F makes with the horizontal is observed to be 30°. Find the magnitude of F. (5)

   (**Total: 10 marks**)

3. A particle of mass 2.5 kg lies at rest at a point with coordinates $(0, 5)$ in the x-y plane, in units of metres. Starting at time $t = 0$, a constant force $\boldsymbol{F}$ of magnitude 5 N and direction $3\mathbf{i} - 4\mathbf{j}$ is applied to the particle. At $t = 5$ s, a second constant force $\boldsymbol{P}$ acts on the particle in order to bring it to instantaneous rest at the origin at $t = 10$ s. Find in $\mathbf{i}, \mathbf{j}$ notation, where appropriate:
   (a) the initial acceleration of the particle; (2)
   (b) the position and velocity of the particle at $t = 5$ s; (3)
   (c) the force $\boldsymbol{P}$; (3)
   (d) the total work done by $\boldsymbol{P}$ between $t = 5$ s and $t = 10$ s. (2)

   (**Total: 10 marks**)

4. A lamina $ABEFD$ is formed by starting off with a square lamina $ABCD$ of uniform density and side $4a$ metres and cutting out a triangle ECF from it, in which $EC = a$ and $CF = 3a$. Forces of magnitude 2 N, 3 N, 4 N, 5 N act on the lamina along the vectors $\overrightarrow{DA}$, $\overrightarrow{BE}$, $\overrightarrow{FD}$, $\overrightarrow{BF}$ respectively. Find:
   (a) the resultant force and torque about the point B; (5)
   (b) the equation of the line along which the resultant force acts. (5)

   (**Total: 10 marks**)

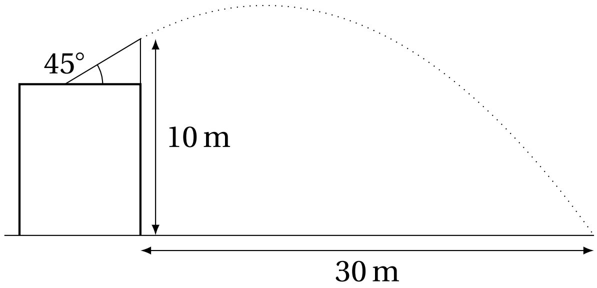
5. A train travels along a track $ABCDE$, where CD represents the track along a passenger station of length 500 m. Between A and B it has a constant top speed of $100\,\mathrm{km\,h^{-1}}$, then it decelerates uniformly for one minute from point B to C at which point it reaches a safe speed of $20\,\mathrm{km\,h^{-1}}$. It keeps this speed until point D when it starts to accelerate uniformly for 1 km until it reaches its original speed of $100\,\mathrm{km\,h^{-1}}$ at the point E. By drawing a velocity-time graph and **not** taking the length of the train into consideration, find:
   (a) the deceleration between B and C in $\mathrm{m\,s^{-2}}$; (2)
   (b) the acceleration between D and E in $\mathrm{m\,s^{-2}}$; (2)
   (c) the total distance from B to E; (2)
   (d) the difference in the total time taken to pass from B to E when compared to the case of the train maintaining a top speed of $100\,\mathrm{km\,h^{-1}}$ throughout BE. (4)

   (**Total: 10 marks**)

6. A toy train of mass m is moving along a track lined up with carriages each having a mass m. The train is initially moving with a velocity u_0 while the carriages are at rest and separated from each other. Strips of velcro are attached to the train and carriages so that they stick together after a collision.
   (a) Determine the speed of the train after the first, second, and third collisions. (6)
   (b) Deduce the speed after the n^{th} collision. (1)
   (c) Express the kinetic energy after the n^{th} collision in terms of the initial kinetic energy. (3)

   (**Total: 10 marks**)

7. A movie stunt involves a car jumping from a ramp placed at the edge of a building and landing at a horizontal distance of 30 m from the foot of the building. The ramp is set at an angle of 45° and the total drop in height is 10 m, as shown below.



(a) Let the initial speed be u_0. Determine the maximum height above the ramp that is attained by the car. (3)

(b) Show that the total time of flight, in seconds, can be expressed as:

$$T = \frac{u_0}{10\sqrt{2}} + \sqrt{\frac{u_0^2}{200} + 2}.$$

(3)

(c) Hence deduce the value of u_0. (4)

(**Total: 10 marks**)

8. A woodcutter wants to drag a log of mass 400 kg a distance of 10 m. To do so, he ties one end of a light rope to the log and the other end to a tractor. The rope is 6.5 m long and is tied to the tractor at a height of 1.6 m above its point of attachment to the log. The log remains horizontal and in contact with the ground while it is being dragged. The coefficient of friction between the log and the ground is $\frac{13}{16}$ and the tractor moves with a constant speed of 6 m s^{-1}.

(a) Determine the tension in the rope. (6)

(b) Find the work done by the tractor in pulling the log a distance of 10 m. (2)

(c) What power needs to be exerted by the tractor just to pull the log? (2)

(**Total: 10 marks**)

9. A fisher wants to cross the Gozo Channel using a boat that can sail at a speed of 40 km h^{-1} in still waters. The channel is 7 km wide and a current of velocity $(-2\sqrt{6}\,\mathbf{i} - 8\,\mathbf{j})$ km h^{-1} is flowing across. Here, **i** is the unit vector directed from the starting position of the fisher to the exact opposite point on the other side, while **j** is a unit vector orthogonal to it.

(a) Determine the angle at which the fisher must sail in order to land exactly at the opposite point on the other side of the channel. (5)

(b) Deduce the resultant velocity of the boat. (3)

(c) Find the time taken to cross the channel, expressing the answer in minutes. (2)

(**Total: 10 marks**)

10. A car of width 1.6 m is driven through a level bend in the road of radius 23.7 m (as measured from the side of the car closest to the centre of the circle) with a speed of $7\,\mathrm{m\,s^{-1}}$. The centre of mass of the car is at its centre.

    (a) Determine the angular acceleration of the car. (3)

    (b) A small book is placed on the back seat. Find the minimum coefficient of friction between the book and the seat so that the book does **not** slide if:

    (i) the book is placed in the middle of the back seat; (4)

    (ii) the book is moved 0.5 m from the middle of the seat towards the inside part of the bend. (3)

(**Total: 10 marks**)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2022 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	3rd May 2022
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions in all.
Each question carries 15 marks.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. Two airplanes A, B are travelling in straight lines with speeds of $204\,\mathrm{km\,h^{-1}}$ and $180\,\mathrm{km\,h^{-1}}$ respectively. At $t = 0$ they are observed to have position vectors $\boldsymbol{r}_A = -11\,\mathbf{i} + 39\,\mathbf{j} + 10\,\mathbf{k}$ and $\boldsymbol{r}_B = 26\,\mathbf{i} - 49\,\mathbf{j} + 3\,\mathbf{k}$, in units of km, relative to the air traffic control tower. They are moving in the directions of $22\,\mathbf{i} - 46\,\mathbf{j} - \mathbf{k}$ and $-5\,\mathbf{i} + 14\,\mathbf{j} + 2\,\mathbf{k}$ respectively.

   (a) Write the vector equations of the straight lines followed by the two airplanes. (4)

   (b) Show that the airplanes would collide and find the time of collision. (5)

   (c) To avert a collision, the pilot of the first airplane A is ordered to immediately lower the aircraft's speed to $153\,\mathrm{km\,h^{-1}}$ at $t = 0$. Find the new minimum distance between the planes. (6)

   (**Total: 15 marks**)

2. A particle is projected from the top of a vertical cliff with a speed of $25\,\mathrm{m\,s^{-1}}$ at an upward angle of θ to the horizontal. The cliff is $200\,\mathrm{m}$ high above the sea. The particle reaches a height h above sea level and splashes in the sea at a horizontal distance d from the cliff foot. Find:

   (a) h and d in terms of θ; (6)

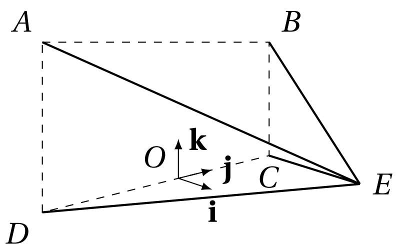
   (b) the maximum values of h and d that can be achieved by the particle. (9)

   (**Total: 15 marks**)

3. A framework consists of four rods in the shape of a pyramid $ABCDE$, where each vertex has position vector given respectively as follows, in units of metres:

$$\boldsymbol{a} = \begin{pmatrix} 0 \\ -12 \\ 8 \end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix} 0 \\ 12 \\ 8 \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix},\ \boldsymbol{d} = \begin{pmatrix} 0 \\ -12 \\ 0 \end{pmatrix},\ \boldsymbol{e} = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}.$$

The rods are placed at AE, BE, CE, and DE, and have masses proportional to their lengths, at 10 kilograms per metre. Apart from the rods' weights acting vertically downwards in the negative $\mathbf{k}$ direction, there are two forces acting on the framework; the first has magnitude 1.7 kN acting along the rod $\overrightarrow{EA}$, while the second has magnitude 1.02 kN and acts along the rod $\overrightarrow{EB}$.



Find:

(a) the resultant force of this system; (3)

(b) the total moment of this system about the origin. (3)

A force $\boldsymbol{F}$ is added to the system so that the resultant force is zero.

(c) Find the force $\boldsymbol{F}$. (4)

(d) If $\boldsymbol{F}$ acts at the point $3\,\mathbf{i}$, determine the additional torque that is needed to bring the system to equilibrium. (5)

(**Total: 15 marks**)

4. A uniform rigid cantilever beam, of length l and weight W, is clamped horizontally at one end and carries at its free end a weight equal to the beam itself.

(a) Find, in terms of W and l, the shearing force and bending moment exerted at:

(i) the clamped end; (2)

(ii) any point along the beam. (4)

(b) Draw diagrams depicting the shearing force and bending moment along the beam. (5)

(c) Deduce from the diagram where the beam is most likely to break. (4)

(**Total: 15 marks**)

5. A particle A of mass m is attached to a fixed point B by a light rigid rod of length a so that it can move in a vertical circle with centre B without friction. The rod is initially vertical with A below B and the particle is then given a horizontal velocity u.

   (a) Find the velocity of the particle in terms of u, g, and a, when A is directly above B. (4)

   (b) When A reaches the point vertically below B after one revolution, it picks up a particle at rest, of the same mass m. Show that the minimum value of u for which the particles together complete another whole circle is $4\sqrt{ga}$. (5)

   (c) If A picks up a particle of mass m each time it completes a circle, find the range of values of u if they are to complete five exact turns. (6)

(**Total: 15 marks**)

6. A sphere A moving with a velocity $10\,\mathbf{i}\,\mathrm{m\,s^{-1}}$ collides with an identical stationary sphere B of equal mass. After the collision, the velocity of sphere B is $(0.7\,\mathbf{i}+2.4\,\mathbf{j})\,\mathrm{m\,s^{-1}}$.

   (a) Show that the direction of motion of sphere A just before the impact is at an angle $\theta = \cos^{-1}(\frac{7}{25})$ with the line joining the centres of the spheres. (2)

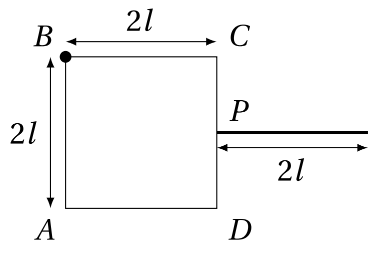
   (b) Determine the magnitude of the velocity of sphere A after the collision. (9)

   (c) What is the percentage kinetic energy lost in the collision? (2)

   (d) Find the coefficient of restitution. (2)

(**Total: 15 marks**)

7. A uniform rod of length $2l$ and mass m is attached to a uniform square lamina $ABCD$ of side lengths $2l$ and mass $2m$ at the midpoint P of CD, aligned in a vertical plane, as in the diagram below. The system is smoothly pivoted at B and is rotated until side BC is horizontal with A below B. It is then released from rest. The moment of inertia of the square $ABCD$ about its centre of mass is $\frac{4}{3}ml^2$.



   (a) Find the moment of inertia of the system about B. (6)

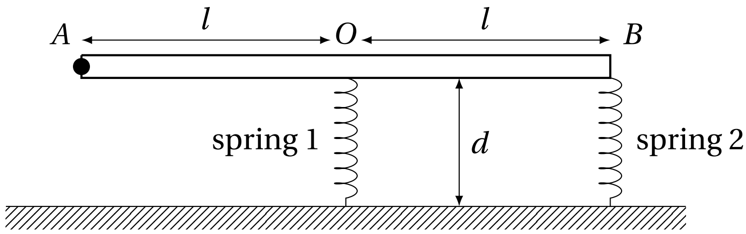
   (b) What is the distance of the centre of mass G of the system from P? (3)

   (c) Find the angular velocity of the combined system when side BC is vertical. (3)

   (d) Determine the periodic time for small oscillations of the combined system. (3)

(**Total: 15 marks**)

8. A uniform rod AB has a mass m and a length $2l$. It is smoothly pivoted from point A at a height d above the floor. Two identical springs of natural length d and modulus of elasticity λ are attached to the rod, one at B and the other at the midpoint O of AB. The other ends of the springs are attached on the floor vertically below B and O respectively when the rod is horizontal. Point B is displaced slightly downwards from the equilibrium position and then released so that the system is set into small vertical oscillations. Assume that in the ensuing motion the springs move in a vertical direction.



(a) Determine the angle θ_0 that the rod subtends with the horizontal when the system is in equilibrium. (5)

(b) Deduce that the oscillatory motion satisfies the equation:

$$\ddot{\theta} + \frac{15\lambda}{4dm}\theta = 0,$$

where θ is measured from the equilibrium position, and each dot represents a derivative with respect to time, t. (6)

(c) Solve the differential equation given that $\theta = \theta_0$ when $t = 0$ s and $\theta = 0.02$ rad when $t = 1$ s. (4)

(**Total: 15 marks**)

9. A particle of mass 4 kg is acted upon by a force of magnitude Av^{-3}, measured in Newtons, where A is a constant and v is the speed, so that it moves in a straight line along the x-axis. At the instant when the time $t = 1$ s the particle is at the origin and has a speed of $v = 3\,\mathrm{m\,s^{-1}}$ while at $t = 16$ s it attains a speed equal to $6\,\mathrm{m\,s^{-1}}$.

(a) Show that the distance travelled with time is given by $x = \frac{12}{5}\left(t^{5/4} - 1\right)$. (8)

(b) (i) Write an expression in terms of t for the power exerted by the force to move the particle. (1)

(ii) Hence determine the work done by the force between $t = 1$ s and $t = 16$ s. (4)

(c) Confirm that the work done is equal to the kinetic energy gained by the particle from $t = 1$ s to $t = 16$ s. (2)

(**Total: 15 marks**)

10. Two uniform disks, each of radius 0.1 m and mass 0.2 kg, are joined at a point P on their circumference so that their centres lie along a straight line passing through P. The combined system rotates freely in a horizontal plane about a smooth pivot at P, with a constant angular velocity of $7\,\mathrm{rad\,s^{-1}}$. At an instant in time, a particle of mass 0.1 kg is dropped at the centre of one of the disks where it sticks.

    (a) Determine the angular velocity of the system after the particle sticks to the disk. (5)

    (b) A retarding couple having moment of magnitude 0.014 N m is then introduced in the system.

        (i) What is the angular acceleration? (2)

        (ii) Find the time taken to bring the system to a halt. (2)

        (iii) Determine the time taken for the first turn after the couple is applied. (4)

        (iv) How many turns will the system make before coming to instantaneous rest? (2)

(Total: 15 marks)