MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2022 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	29 August 2022
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions is 100.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A particle of mass 20 kg lies on a rough plane inclined at an angle of 30° to the horizontal. The coefficient of friction between the particle and the plane is 0.3. A horizontal force of magnitude F is applied to the particle, towards the plane, as it slides down the plane with constant speed along a line of greatest slope. Find:

   (a) the value of F; (6)

   (b) the normal reaction in Newtons between the particle and the plane. (4)

   (**Total: 10 marks**)

2. A particle of mass 5 kg is suspended vertically from a fixed point O by a light elastic string of natural length 40 cm and unknown modulus of elasticity λ. At equilibrium, the string extends by a length of 10 cm.

   (a) Find the modulus of elasticity λ. (2)

   (b) Find the elastic energy stored in the string at equilibrium. (2)

   (c) The particle is now pulled down and the string is stretched further by 20 cm. The particle is then released. Find the maximum height reached by the particle from the point of release. (6)

   (**Total: 10 marks**)

3. A uniform rod AB of length $4l$ and mass m is suspended from a point O by two strings AO and BO with fixed lengths $5l$ and $3l$, respectively. Find:

   (a) the angle $\widehat{AOB}$; (2)

   (b) the tensions in the two strings, in terms of m; (6)

   (c) the magnitude and direction of the reaction at the point O. (2)

(**Total: 10 marks**)

4. Two particles of masses 2 kg and 3 kg are connected by a light inelastic string passing over a smooth fixed pulley. The masses are released from rest with the heavier one initially at a height of 1 m above the ground. The 3 kg mass hits the ground and remains motionless. Find:

   (a) the tension in the string before the mass hits the ground; (3)

   (b) the time taken for the heavier particle to hit the ground; (2)

   (c) the time during which the string is slack. (5)

(**Total: 10 marks**)

5. Three forces act on a square $ABCD$ of side 1 m as follows:

   $\boldsymbol{F}_1 = (a\mathbf{i} - 3\mathbf{j})$N acts at point A,
   $\boldsymbol{F}_2 = (5\mathbf{i} + b\mathbf{j})$N acts at point B,
   $\boldsymbol{F}_3 = (2\mathbf{i} + 10\mathbf{j})$N acts at a point M on CD,

   where $\mathbf{i}$ is the vector $\overrightarrow{DC}$ and $\mathbf{j}$ is the vector $\overrightarrow{DA}$.

   (a) If the forces are in equilibrium, find the values of the constants a and b, and the vector $\overrightarrow{DM}$. (5)

   Forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are now swapped so that $\boldsymbol{F}_1$ acts at B and $\boldsymbol{F}_2$ acts at A.

   (b) Calculate the magnitude and direction of the resultant force. (2)

   (c) Find the moment of the resultant couple about the point D. (3)

(**Total: 10 marks**)

6. Two identical billiard balls, A and B, of equal mass lie at rest on a smooth surface. A third identical billiard ball C, of the same mass, is launched directly towards A along the line AB with a speed of $8\,\mathrm{m\,s^{-1}}$. The coefficient of restitution between any two balls is $\frac{1}{2}$.

   (a) Determine the velocities of A and C after they collide. (3)

   (b) With the velocity attained, A now collides with B. Determine their velocities after the impact. (3)

   (c) Billiard balls A and C collide again. Find their velocities after the third impact. (3)

   (d) Explain why there are **no** further collisions between the billiard balls. (1)

(**Total: 10 marks**)

7. A basketball player is practising free throws. The ball is being thrown from a height of 2 m and is aimed at the centre of the backboard in such a way that it bounces downwards into the hoop. The centre of the backboard is at a height of 3.6 m from the ground and at a horizontal distance of 4.6 m from the player. Ignore the size of the basketball and assume that its impact on the backboard is elastic.

   (a) Explain why it is important for the basketball to hit the centre of the board when it is moving along its downward trajectory rather than its upward trajectory. (1)

   (b) If the initial vertical velocity component of the ball is $6\,\mathrm{m\,s^{-1}}$, determine the time taken for the ball to hit the centre of the backboard. (5)

   (c) Find the initial horizontal velocity component. (2)

   (d) At what angle to the horizontal does the player throw the basketball? (2)

   (**Total: 10 marks**)

8. A car of mass 1500 kg attains a maximum constant speed of $20\,\mathrm{m\,s^{-1}}$ when going down an incline with its engine switched off. The maximum constant speed that can be attained when going up the same incline with the engine switched on is $28\,\mathrm{m\,s^{-1}}$. The incline makes an angle of θ with the horizontal, where $\theta = \sin^{-1}(\frac{1}{30})$, while throughout the motion the car experiences a resistive force having a magnitude of the form $F = kv^2$ with v being the speed and k a constant.

   (a) Show that $k = \frac{5}{4}\,\mathrm{N\,s^2\,m^{-2}}$. (3)

   (b) Find the power output of the engine at the speed of $28\,\mathrm{m\,s^{-1}}$ up the incline. (4)

   (c) What is the maximum speed attained by the car if it were to move on a horizontal road, with the same engine power as in part (b)? (3)

   (**Total: 10 marks**)

9. Two planes are flying at the same altitude. At $t = 0$ plane A is at position $(-15\mathbf{i} + 50\mathbf{j})\,\mathrm{km}$ travelling with a uniform velocity of $(300\mathbf{i} + 200\mathbf{j})\,\mathrm{km\,h^{-1}}$, while plane B is at position $-20\mathbf{j}\,\mathrm{km}$ travelling with a uniform velocity of $500\mathbf{j}\,\mathrm{km\,h^{-1}}$.

   (a) Find the velocity of plane A relative to plane B. (1)

   (b) Show that when $t = 3\,\mathrm{min}$ the position of plane A has **no** component in the $\mathbf{i}$ direction. (2)

   (c) Determine the distance of closest approach of the two planes. (4)

   (d) At what time is the distance of closest approach attained? (3)

   (**Total: 10 marks**)

Questions continue on the next page.

10. A light rough rod OA is smoothly pivoted at O so that the end A rotates in a horizontal plane. A ring of mass m is threaded on the rod and the coefficient of friction between the rod and the ring is μ. When the system rotates at a constant angular speed ω at a fixed angle θ to the downward vertical, the ring reaches a distance l from the pivot.

(a) Show that the frictional force F and the normal reaction N at the point of contact between the ring and the rod are given by:

$$F = mg\cos\theta + m\omega^2 l \sin^2\theta,$$

$$N = m(g - \omega^2 l \cos\theta)\sin\theta,$$

assuming that both are positive in an upwards direction. (8)

(b) Deduce the maximum value for the distance l. (2)

(**Total: 10 marks**)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2022 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	30 August 2022
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions in all.
Each question carries 15 marks.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A particle A of mass 2 kg is acted upon by a constant force of 20 N in the direction of $-3\mathbf{i}+4\mathbf{j}$. Initially it has position vector $4\mathbf{j}$ and is moving at a speed of $9\sqrt{2}\,\mathrm{m\,s^{-1}}$ in the direction of $\mathbf{i}-\mathbf{j}$. Find:

   (a) the position vector of the particle at time t; (4)

   (b) the work done by the force in the first 5 s. (5)

   Another particle B is moving in a straight line with position vector $\mathbf{r}(t) = -3\mathbf{i}+8\mathbf{j}+t(5\mathbf{i}-12\mathbf{j})$ at time t. Find:

   (c) the position vector of A relative to B at time t; (2)

   (d) the value of t when the particles have minimum distance between them. (4)

   (**Total: 15 marks**)

2. A uniform rigid horizontal beam $ABCD$ of length 10 m and mass 34 kg, is simply supported at B and C, where $AB = 1$ m and $CD = 2$ m. A weight of 40 N is attached at the midpoint M of CD.

   (a) Calculate the forces acting at the support points B and C. (2)

   (b) Draw the shearing force and bending moment diagrams of the beam for this load. (8)

   (c) Find the maximum magnitude of the bending moment and the corresponding position on the beam. (5)

   (**Total: 15 marks**)

3. Two cars R and S compete in a drag race along a straight road $ABCDEFG$. Both cars start off at A but R accelerates uniformly at $3g$ to reach a maximum speed of $486\,\mathrm{km\,h^{-1}}$ at point B, while S accelerates uniformly at $2.9g$ to reach $504\,\mathrm{km\,h^{-1}}$ at point C. They continue at their top speed until the finish line at D, where $AD = 400\,\mathrm{m}$. At this point they both decelerate uniformly at $20\,\mathrm{m\,s^{-2}}$ with R and S stopping at F and G respectively.

   (a) Draw the two velocity-time graphs for R and S on the same diagram. (5)

   (b) Determine who wins the race and the time difference between the cars as they cross the finish line D. (3)

   (c) Calculate the distance DE where E is the place at which S overtakes R. (4)

   (d) Find the distance FG at the end of the race. (3)

   (**Total: 15 marks**)

4. At time $t = 0$, a particle is projected upward from the bottom of a cliff of height h, with speed v and angle of inclination $60°$ with the horizontal. At time $t = T$, another particle is projected downward from the top of the cliff with the same speed and angle of inclination $-30°$. If the two particles collide, show that:

   (a) they collide at time $\frac{\sqrt{3}}{\sqrt{3}-1}T$; (3)

   (b) $h = \frac{2}{\sqrt{3}-1}vT - \frac{1}{(\sqrt{3}-1)^2}gT^2$; (7)

   (c) if the particles collide at ground level then $gT = (\sqrt{3}-1)v$. (5)

   (**Total: 15 marks**)

5. A particle P of mass m is attached to two fixed points A and B by light inelastic strings AP and BP of length $4l$ and $3l$ respectively. The point A is directly above B with $AB = 3l$. The particle moves in a horizontal circle at constant angular speed ω such that both strings are taut.

   (a) If the tensions in the strings are equal, find an expression for ω in terms of l and g. (6)

   (b) Show that the angular speed at which string BP just becomes slack is $\sqrt{\frac{5}{14}}\omega$. (3)

   (c) If the endpoints of the strings at A and B are swapped, and the angular speed tripled to 3ω, what is the ratio of the tensions in the two strings? (6)

   (**Total: 15 marks**)

6. A smooth sphere A collides with an identical sphere B which is at rest. Just before the collision, the velocity of sphere A makes an angle of $30°$ with the line joining the centres of mass of the two spheres, while its velocity just after the collision makes an angle of $60°$ with this line.

   (a) Show that $v_A = \frac{1}{\sqrt{3}}u_A$, where u_A and v_A represent the speed of sphere A before and after the collision respectively. (4)

   (b) Find the speed of sphere B after the collision in terms of u_A. (6)

   (c) What percentage of the total kinetic energy is lost in the collision? (3)

(d) Determine the coefficient of restitution. (2)

(**Total: 15 marks**)

7. A particle P of mass m is placed on the inner smooth surface of a static cylindrical tube having its axis of symmetry oriented in the horizontal direction. Once released from rest, the particle is observed to oscillate in a plane perpendicular to the axis of symmetry of the cylindrical tube. The distance from the centre of the cylindrical tube O to the inner surface is r.

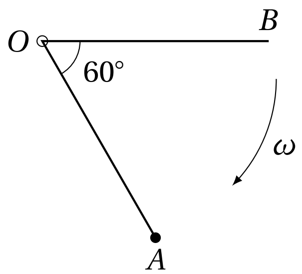
   (a) Show that the general equation of motion is given by:
   $$\ddot{\theta} + \frac{g}{r}\sin\theta = 0$$
   where θ is the angle that OP makes with the downward vertical and each dot represents a derivative with respect to time. Hence, deduce the frequency of small oscillations. (6)

   (b) Use the equation obtained in part (a) to derive the following relation:
   $$\dot{\theta}^2 = \frac{2g\cos\theta}{r}$$
   given that when $\theta = 0$ the angular speed is $\dot{\theta} = \sqrt{2g/r}$. (6)

   (c) Find a general expression for the reaction force acting on the particle, in terms of m, g, and θ. (3)

(**Total: 15 marks**)

8. A uniform rod AOB of length $2l$ and mass $4m$ is bent at its centre O such that $\widehat{AOB} = 60°$. A particle of mass m is attached at the end A, while the combined system is smoothly pivoted at O so that it can rotate in a vertical plane containing O.



   (a) Determine the moment of inertia of the combined system. (2)

   (b) The side OB is held horizontally with A below B, as shown in the diagram, and the system is given an angular velocity ω directed downwards. Find an expression for the minimum value of ω required for A to reach a point directly above O. (9)

   (c) What is the period for small oscillations of the system? (4)

(**Total: 15 marks**)

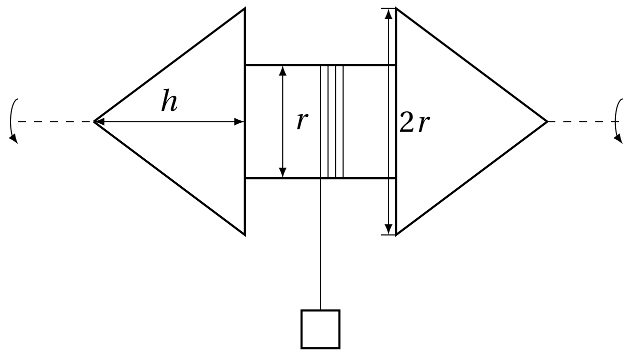
9. A uniform rod of mass $3m$ and length l lies at rest on a smooth horizontal plane. It is smoothly pivoted at one of its ends so that it can rotate freely about it. A particle of mass m, moving with a speed u_0 along the plane in a direction perpendicular to the rod, strikes it at a distance x from the pivot point and rebounds from it. The impact is perfectly elastic.

   (a) Show that the angular velocity of the rod immediately after the collision is:
   $$\omega = \frac{2u_0 x}{l^2 + x^2}.$$
   (7)

   (b) Determine the value of x for which ω has a stationary point and show that this point is a maximum. (8)

   (**Total: 15 marks**)

10. A uniform solid cylinder of mass $\frac{8}{5}m$ and radius $\frac{1}{2}r$, has a right circular smooth uniform solid cone, having mass m, base radius r, and height h, attached at each end at the base so that the axis of symmetry of the three solids are aligned. One end of a light thin inextensible cord is wound a few times around the middle of the cylinder, while the other end is attached to a block of mass $\frac{9}{5}m$. The combined system is then smoothly pivoted at the tips of the cone so that it can rotate freely with the axis of symmetry aligned horizontally while the block hangs underneath with the cord taut.



   (a) Show that the moment of inertia of one cone about its axis of symmetry is $\frac{3}{10}mr^2$. (8)

   (b) The block is allowed to fall freely under its own weight with the cord unwinding without slipping while at the same time inducing the combined system to rotate.

      (i) Given that the moment of inertia of a cylinder of mass M and radius R is $\frac{1}{2}MR^2$, obtain an expression for the moment of inertia of the combined system. (2)

      (ii) Determine the linear velocity and linear acceleration of the block at the instant that the cord unwinds by a short length l. (5)

   (**Total: 15 marks**)