MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2023 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	29^{th} April 2023
TIME:	09:00 a.m. to 12:05 p.m.

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions is 100.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A mass of 3 kg lies on a rough plane inclined at 30° to the horizontal. A horizontal force P of magnitude 30 N holds the mass in limiting equilibrium by pressing it against the plane. The force acts in a vertical plane containing a line of greatest slope of the inclined plane and the object is on the point of moving up the plane. Find:

   (a) the magnitude of the normal reaction; (5)

   (b) the coefficient of friction between the object and the plane. (5)

   (**Total: 10 marks**)

2. An airplane lands at point A on a runway at a speed of $126\,\mathrm{km\,h^{-1}}$. As soon as it touches the runway, it decelerates uniformly up to point B on the runway, when it reaches a safe speed of $18\,\mathrm{km\,h^{-1}}$ in 30 s. After this point it continues 'taxiing' at a constant speed until it reaches its destination C which is 3 km away from the landing spot.

   (a) Draw a velocity-time graph, using units of $\mathrm{m\,s^{-1}}$ for the velocity. (3)

   (b) Find the distance AB. (2)

   (c) Find the time taken between A and C. (2)

   (d) A larger plane lands at point A with a speed of $234\,\mathrm{km\,h^{-1}}$. How long does it take to reach the destination with the same deceleration and taxiing speed? (3)

   (**Total: 10 marks**)

3. Two beacons A and B have position vectors $(-24, -20)$ and $(30, 10)$ respectively, relative to a Cartesian coordinate system where the x- and y-axes point East and North, respectively, and the unit of length is the km. A ship is moving with a constant speed of $40\,\mathrm{km\,h^{-1}}$ at an angle of $20°$ anticlockwise from the x-axis. At 6 a.m., the ship is observed to have a position of $(1, 0.9)$ relative to beacon A.

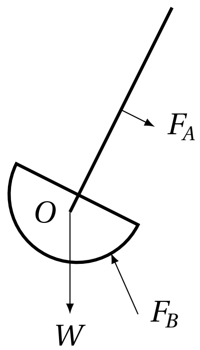
   (a) Express the velocity in **i**, **j**, notation. (2)

   (b) Find the vector equation of the line followed by the ship. (3)

   (c) Find the time when the ship is due south of beacon B, and its distance from it. (5)

   (**Total: 10 marks**)

4. A sailing boat weighing 4 tons has a mast of height 10 m from the centre of mass of the boat. At equilibrium, it has three forces acting on it in a vertical plane as in the diagram below: the weight W, a wind force F_A of magnitude 10 kN acting perpendicular to the mast at its midpoint; and a buoyancy force F_B. The mast makes an angle of $20°$ to the vertical. Use a coordinate system in which the centre of mass is at the origin, the x-axis points horizontally to the right and the y-axis is vertical.



   (a) Find the magnitude of F_B and its line of action. (5)

   (b) The wind increases in strength so that $F_A = 20\,\mathrm{kN}$. If, at equilibrium, the magnitude of F_B increases by 25%, find the new angle that the mast makes with the vertical. (5)

   (**Total: 10 marks**)

5. A spring with modulus of elasticity $20\,\mathrm{N\,m^{-1}}$ and natural length 10 cm, is hung from a nail. An object with a mass of 5 kg is attached to the spring and allowed to come to rest. The object is then pulled down a further 25 cm and released so that the system performs undamped vertical oscillations.

   (a) Find:

      (i) the maximum acceleration of the object; (3)

      (ii) the period of oscillation; (3)

      (iii) the maximum resultant force acting on the particle. (3)

   (b) Which of the answers in part (a) would change if the spring is extended by only 10 cm below the equilibrium point? (1)

   (**Total: 10 marks**)

6. A smooth sphere A of mass m moves to the right with speed u_0. A second sphere B, of equal radius and mass $2m$, moves to the left with speed $u - u_0$, where $u \geqslant u_0$. The two spheres collide directly, with a coefficient of restitution equal to e.

   (a) Show that the speed of A after the collision is $u_0 - \frac{2}{3}(1+e)u$ while that of B is $u_0 - \frac{1}{3}(2-e)u$. (6)

   (b) Find an expression for the value of e in terms of u and u_0, for which B is stationary after the collision. (1)

   (c) Hence find the range of possible values of u/u_0. (3)

   (**Total: 10 marks**)

7. An escalator is used to lift a person of weight 65 kg up a floor of height 2.5 m, at constant velocity. The escalator is inclined at 30° to the horizontal and the vertical component of the velocity is $0.5\,\mathrm{m\,s^{-1}}$. Any resistive forces can be ignored. Determine:

   (a) the potential energy gained by the person when going up one floor; (2)

   (b) the work done in moving the person in a horizontal direction; (1)

   (c) the horizontal velocity; (3)

   (d) the kinetic energy of the person during the motion; (2)

   (e) the power used to move the person from one floor to another. (2)

   (**Total: 10 marks**)

8. In a rugby match, player A has the ball and is moving uniformly with a speed of $11.6\,\mathrm{m\,s^{-1}}$ at an angle $\alpha = \cos^{-1}(20/29)$ to the x-axis. Player B initially has position $46.8\,\mathbf{j}$ relative to player A and is moving in a straight line with a speed of $17\,\mathrm{m\,s^{-1}}$ to intercept player A. Let the velocity of B be $u\,\mathbf{i} + v\,\mathbf{j}$.

   (a) Express the velocity of player A in terms of $\mathbf{i}$ and $\mathbf{j}$. (2)

   (b) Determine u and v. (5)

   (c) Find the time taken by B to intercept A. (3)

   (**Total: 10 marks**)

9. A jet airplane of mass 24 tons is flying on a horizontal circular path of radius 10 km with a constant speed v while waiting to land at Luqa airport. In order to keep the airplane in its circular path, the pilot banks it at an angle $\alpha = \sin^{-1}(7/25)$ to the horizontal. The lifting force F is perpendicular to the wings. Find:

   (a) the lifting force F; (3)

   (b) the centripetal acceleration; (3)

   (c) the value of v; (2)

   (d) the time taken for the jet to go once round the circular path. (2)

   (**Total: 10 marks**)

10. A particle is projected from the point $(0,0)$ with a horizontal velocity component of $2\,\mathrm{m\,s^{-1}}$. In the subsequent motion, its vertical displacement y is related to the horizontal displacement x by the relation:

$$y = \sqrt{3}\,x - \frac{5x^2}{4}.$$

(a) Obtain an equation for y in terms of the time t. (2)

(b) Deduce the value of the initial vertical velocity component. (2)

(c) Determine the angle to the horizontal with which the particle is projected. (2)

Considering the motion of the particle until it reaches $y = 0$ again, find:

(d) the horizontal distance travelled; (2)

(e) the time of flight. (2)

(**Total: 10 marks**)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2023 FIRST SESSION

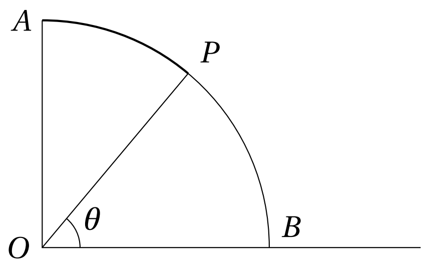
SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	2nd May 2023
TIME:	4:00 p.m. to 7:05 p.m.

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions in all.
Each question carries 15 marks.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A water slide is in the shape of a vertical quarter-circle OAB of radius 10 m as in the diagram below. People slide from rest from the top point A until they lose contact with the slide and reach the pool water below in free fall. A person P is to be modelled as a point particle of mass m and friction is to be neglected.



(a) Show that the speed of P, while it remains in contact with the slide, is given by:

$$v^2 = 200(1 - \sin\theta),$$

where θ is the angle that OP makes with the horizontal. (2)

(b) Find an expression for the normal reaction on P in terms of m, θ and v. (4)

(c) Hence find the angle θ when P loses contact with the slide. (3)

(d) What is the height above OB and the speed of P at this point? (2)

(e) Determine the horizontal distance between the foot of the slide and the point where P touches OB extended. (4)

(**Total: 15 marks**)

2. A football player would like to kick the ball so that it reaches a point P just within the goalposts, where P is 2.3 m above ground and 30 m distant. The player is able to kick the ball at $21\,\mathrm{m\,s^{-1}}$.

   (a) Neglecting air resistance, what is the smallest angle to the horizontal that the ball should be kicked at to reach its destination? (8)

   (b) How long does it take for the ball to reach point P? (2)

   A second player, who is distant 5 m from point P, wants to reach the same spot.

   (c) If the second player starts running from rest, 0.5 s after the ball is kicked, how much would the player need to accelerate uniformly to reach point P at the moment that the ball reaches it? (5)

   (**Total: 15 marks**)

3. Two particles A and B are connected together by a light inextensible string that passes over a light smooth pulley at P which is attached at the top of a plane inclined at 25° to the horizontal. Particle A has mass 0.2 kg and is at rest at the foot of the plane, with AP equal to 1 m. Particle B has mass 0.3 kg and hangs freely. The coefficient of friction between particle A and the plane is 0.4. Find:

   (a) the frictional force acting on A; (4)

   (b) the acceleration of the particles; (3)

   (c) the time taken for A to reach the pulley. (3)

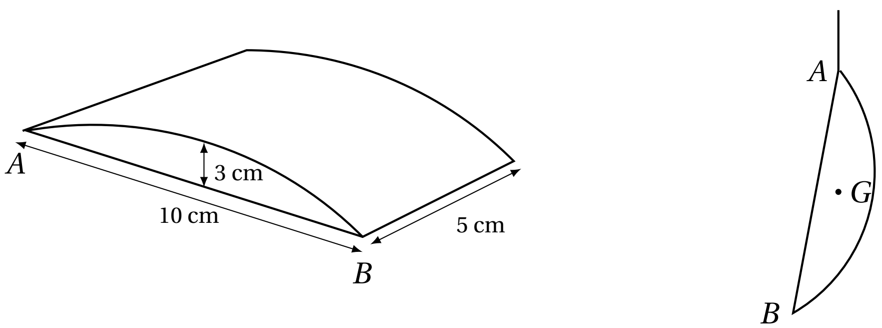
   When A reaches point P, it is brought to rest and the string is cut.

   (d) Find how long it takes for A to reach its original position from point P. (3)

   (e) Name **TWO** considerations that need to be taken into account to make the mathematical model more realistic. (2)

   (**Total: 15 marks**)

4. A computer mouse is modelled as a slice off a solid right cylinder of uniform density as shown in the diagram below. The dimensions of the mouse are 10 cm by 5 cm by 3 cm.



   (a) Derive the position of the centre of mass for a uniform circular **sector** of radius r and central angle 2θ. (4)

   (b) Hence, find the height of the centre of mass of the mouse from its flat surface. (5)

   (c) Show that the segment cross-section of the mouse is part of a sector of radius $r = \frac{17}{3}$ cm and central angle $2\theta = 2\arcsin\frac{15}{17}$. (3)

   (d) If the mouse is hung freely from the midpoint of the shorter side by a straight wire, what is the angle between AB and the vertical? (3)

   (**Total: 15 marks**)

5. (a) A particle of mass 1 kg moves so that the force acting on it is $\boldsymbol{F} = 4\boldsymbol{r}$, where $\boldsymbol{r}$ is the position of the particle at time t. Show that:

   (i) $\boldsymbol{r} = e^{2t}\boldsymbol{a} + e^{-2t}\boldsymbol{b}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ are constant vectors, satisfies this equation; (3)

   (ii) $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ is independent of t. (3)

   (b) A particle A moves in a plane along a parabola with vector equation:

$$\boldsymbol{r} = \begin{pmatrix} 1+t \\ 2+3t-t^2 \end{pmatrix}.$$

   A second particle B moves along a straight line with algebraic equation $y = 3 - x$. They both start together at one point of intersection of the parabola and straight line and arrive simultaneously at the other point of intersection.

   (i) Find the coordinates of the two points of intersection. (4)

   (ii) Write down the position vector of B if it accelerates uniformly along the straight line. (5)

(Total: 15 marks)

6. A mass m is connected to two vertical springs AB and BC at the point B, while points A and C are fixed with A directly above C and distant $2l$ from it. Both springs have a natural length l; the modulus of elasticity of AB is 2λ and that of BC is λ. The displacement of m is along a vertical line.

   (a) Find the vertical displacement of B from the midpoint of AC, at equilibrium, in terms of l, g, m, and λ. (3)

   The mass is now displaced vertically downwards a distance D from the equilibrium point and released from rest so that it undergoes vertical oscillations.

   (b) Show that the equation of motion for the mass is given by $\ddot{x} + \frac{3\lambda}{ml}x = 0$, where x is the vertical displacement from the equilibrium position and each dot represents a derivative with respect to time. (5)

   (c) Obtain an expression for the frequency of oscillations f in terms of the given variables. (1)

   (d) Solve the differential equation shown in (b). (6)

(Total: 15 marks)

7. A smooth sphere A of mass 5 kg collides with a stationary smooth sphere B of the same radius, at an angle of 45° to the line joining their centres. After the collision, the two spheres move at right angles to each other. Initially A is moving at a speed of $2\sqrt{2}\,\text{m}\,\text{s}^{-1}$ and the coefficient of restitution between the spheres is $\frac{1}{2}$.

   (a) Find the components of the final velocities of A and B parallel to the line joining the centres of the spheres and perpendicular to this direction. (9)

   (b) Determine the mass of B. (3)

   (c) Calculate the percentage decrease in the total kinetic energy. (3)

(Total: 15 marks)

8. A uniform disc of mass m and radius r rolls without slipping down a rough slope inclined at an angle $\phi = \cos^{-1}(63/65)$ to the horizontal, between A and B, which lie along a line of greatest slope with $AB = 487.5\,\text{cm}$. The disc rotates about an axis through its centre perpendicular to its face.

   (a) Show that the moment of inertia of the disc is $\frac{1}{2}mr^2$. (5)

   (b) If the disc is initially at rest at A, determine the linear speed when it reaches B. (5)

   (c) Find the time taken to travel from A to B. (3)

   (d) Calculate the linear acceleration of the disc. (2)

   (**Total: 15 marks**)

9. A uniform rigid rod AB of length $5r$ and mass m is fixed at B perpendicularly to the surface of a uniform solid sphere of mass $\frac{5}{3}m$ and radius r. The combined system is then freely pivoted from the point A.

   (a) Find the moment of inertia of the system about A in terms of the given variables. (5)

   (b) Obtain an expression for the distance between the centre of mass G of the combined system and A. (3)

   (c) Hence, or otherwise, deduce the period of small oscillations for the combined system. (2)

   (d) The combined system is now released from rest at an angle θ_0 to the downward vertical. Show that the magnitude of the angular velocity ω when the system makes an angle θ with the downward vertical is:
   $$\omega = 5\sqrt{\frac{\cos\theta - \cos\theta_0}{69r}}.$$ (5)

   (**Total: 15 marks**)

10. A square lamina of side length $3\sqrt{2}\,\text{m}$ and mass $6\,\text{kg}$ is pivoted at its centre of mass on a flat horizontal surface. An impulse J is given to the lamina at one of the corners in a direction that is perpendicular to a diagonal so that it starts rotating with an angular velocity of $5\,\text{rad}\,\text{s}^{-1}$. There are no frictional forces.

    (a) Determine the magnitude of the applied impulse. Note that the moment of inertia of a rectangular lamina of mass M and side lengths L and B about an axis through the centre of mass and perpendicular to the lamina is given by $\frac{1}{12}M(L^2 + B^2)$. (5)

    (b) Calculate the kinetic energy gained by the lamina. (2)

    A constant retarding couple is now applied to the lamina so as to slow it down to an angular velocity of $1\,\text{rad}\,\text{s}^{-1}$ after completing exactly 9 complete turns. Find:

    (c) the amount of kinetic energy lost by the lamina; (3)

    (d) the magnitude of the moment of the couple; (3)

    (e) the angular deceleration of the lamina. (2)

    (**Total: 15 marks**)