**MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD**
**UNIVERSITY OF MALTA, MSIDA**

**MATRICULATION CERTIFICATE EXAMINATION**
**ADVANCED LEVEL**
**SEPTEMBER 2013**

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| **SUBJECT**: | PURE MATHEMATICS |
| **PAPER NUMBER**: | I |
| **DATE**: | 3rd SEPTEMBER 2013 |
| **TIME**: | 9.00 a.m. to 12.00 noon |

**Directions to Candidates**

*Answer* **ALL** *questions.*

*Each question carries* 10 *marks.*

*Graphical calculators are* **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. The points $A$ and $B$ have position vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $6\mathbf{i}+\mathbf{j}-3\mathbf{k}$, respectively.
   (a) Find the vector equation of the line $\ell_1$ that passes through both $A$ and $B$.
   **[2 marks]**
   (b) The point $C$ has position vector $2\mathbf{i}+3\mathbf{j}+\alpha\mathbf{k}$. Find $\alpha$ given that $\angle ACB$ is a right angle and the distance from $A$ to $C$ is $\sqrt{3}$.
   **[5 marks]**
   (c) Find the other two angles of the triangle $ABC$.
   **[3 marks]**

2. (a) Find the gradient function $\frac{dy}{dx}$ of the curves given by the following equations:
   (i) $x^2e^{x+y} = \log_{10}(3x)$;
   (ii) $y = 6t^2$ and $x = \sec t$.
   **[3, 2 marks]**
   (b) Differentiate $\sqrt{(2x+1)^{\frac{3}{2}} + (2x+1)^{\frac{1}{2}}}$ with respect to $x$, giving your answer in simplified form.
   **[5 marks]**

3. A point $P$ in a plane moves such that it remains at a fixed distance $r$ from a fixed point $A$ with coordinates $(r, r)$.
   (i) Find the equation of the locus of point $P$ (in terms of $r$).

   Another point $Q$ in the same plane moves such that it is equidistant from the point $A$ and the line $y = c$, for some constant $c \neq r$.
   (ii) Find the equation of the locus of point $Q$ (in terms of $r$ and $c$).
   (iii) If the locus of $P$ and that of $Q$ intersect at exactly one point, find the two possible values of $c$ (in terms of $r$).
   (iv) For each of the two values of $c$ found in (iii) above, deduce the coordinates of the respective point of intersection (in terms of $r$).

**[1, 1, 6, 2 marks]**

4. Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{x^2 + 1},$$
given that $y = 2$ when $x = 0$.

**[10 marks]**

5. (a) Solve the following simultaneous equations:
$$y = \log_2 x^2,$$
$$\frac{1}{2}(y + 1) = \log_4 8x.$$
   (b) For what range of values of $k$ does the equation
$$kx^2 - (k+4)x + 4 = 3x^2 + k^2 + 3k$$
   have real roots?

**[5, 5 marks]**

6. The function $f$ is defined for all real values of $x$ by
$$f(x) = |x| + |x - 3|.$$
   (a) For values of $x$ such that $x < 0$, show that $f(x) = 3 - 2x$.
   (b) Write down expressions for $f(x)$ in a form not involving modulus signs for each of the intervals:
$$\text{(i)}\ \ x > 3; \qquad \text{(ii)}\ \ 0 \le x \le 3.$$
   (c) Sketch the graph of $f$ and write down the equation of its line of symmetry.
   (d) State the range of $f$ and solve the equation $f(x) = 4$.

**[2, 3, 2, 3 marks]**

7. (a) Out of a class of students, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

**[5 marks]**

(b) A word is defined as a nonempty (possibly meaningless) sequence of letters. How many 6-word sentences can be made using each of the 26 letters of the alphabet exactly once? Generalize the result to the number of sentences consisting of $w$ nonempty words using exactly once each letter from a $\ell$-letter alphabet.

**[5 marks]**

8. Let

$$Q(x) = \frac{2x^2 - 3x - 11}{x^2 - 2x - 3}.$$

(a) Express $Q(x)$ in partial fractions and show that when $|x| < 1$ then

$$Q(x) = 2 + \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n + \frac{3}{2}\sum_{n=0}^{\infty}(-x)^n.$$

(b) Deduce that

$$\sum_{n=0}^{\infty}\frac{1}{6\times 9^n} + \frac{3}{2\times(-3)^n} = \frac{21}{16}.$$

**[6, 4 marks]**

9. (a) Express in partial fractions

$$\frac{1}{x(x+1)(x+2)}.$$

Hence show that

$$\int_1^n \frac{1}{x(x+1)(x+2)}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{4n(n+2)}{3(n+1)^2}\right).$$

**[6 marks]**

(b) Use integration by parts to find

$$\int x^3 \sin x \,\mathrm{d}x.$$

**[4 marks]**

10. The first and second terms of a geometric progression are $\cos\theta$ and $\sin\theta$ respectively.
    (a) Find the common ratio $r$ and the third term of this geometric progression in terms of $\theta$.
    **[2 marks]**
    (b) Given that the first and third terms of this geometric progression and $\tan\theta$ are three consecutive terms of an arithmetic progression, find the general solution for $\theta$. Give your answer in radians correct to *two* decimal places.
    **[5 marks]**
    (c) Suppose that $0 < r < 1$. Find the sum to infinity of the geometric progression in this case.
    **[3 marks]**

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**ADVANCED LEVEL**
**SEPTEMBER 2013**

| | |
|---|---|
| **SUBJECT**: | PURE MATHEMATICS |
| **PAPER NUMBER**: | II |
| **DATE**: | 4th SEPTEMBER 2013 |
| **TIME**: | 9.00 a.m. to 12.00 noon |

**Directions to Candidates**

*Answer* **SEVEN** *questions. Each question carries* 15 *marks.*

*Graphical calculators are* **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}\cos x + y\sin x = \frac{x(\cos x)^2}{x^2+1},$$

given that $y = 7$ when $x = 0$.

**[7 marks]**

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \sin x,$$

given that $y = -\frac{4}{25}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{28}{25}$ when $x = 0$.

**[8 marks]**

2. The lines $\ell_1$ and $\ell_2$ have vector equations

$$\mathbf{r}_1 = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k});$$
$$\mathbf{r}_2 = 7\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

(a) Find the angle between the two lines. Find the point $A$ where the two lines intersect.
(b) Find the equation of the plane $\Pi_1$ that contains both $\ell_1$ and $\ell_2$.
(c) Find the equation of the plane $\Pi_2$ that contains $\ell_1$ and is perpendicular to $\Pi_1$.
(d) Verify that the point $C$ with position vector $\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ is on $\Pi_1$. Find the distance from $C$ to $\Pi_2$.

**[5, 4, 4, 2 marks]**

3. The function $f$ is given by

$$f(x) \equiv \frac{(x-2)(x+1)}{x-3}.$$

(i) Sketch the graph of the curve $y = f(x)$, showing clearly the coordinates of the points where the curve cuts the coordinate axes, the coordinates of any stationary points, and the equations of the linear asymptotes.

(ii) The part of the curve $y = f(x)$ between $x = 4$ and $x = 7$ is rotated by one complete revolution about the $x$-axis to form a solid of revolution. Find the volume of the solid formed.

**[10, 5 marks]**

4. (Note: throughout this question, all angles have to be taken in radians.)

(a) Show that the equation $e^{\sin x} = x$ has a solution between 2 and 2.5. Use the Newton-Raphson method to find an approximate value of this solution, taking 2.5 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

**[7 marks]**

(b) Evaluate the integral $\displaystyle\int_{-1}^{1} e^{\sin \frac{x}{5}}\, \mathrm{d}x$

(i) by Simpson's Rule with an interval width of $h = 0.5$, and

(ii) by integrating the *first three terms* of the series expansion of $e^{\sin \frac{x}{5}}$. [One may assume that the first *three* terms in the series expansion of $e^{\sin \frac{x}{5}}$ are the same as the first *three* terms in the series expansion of $e^{\frac{x}{5}}$.]

Give your answers to *four* decimal places.

**[4, 4 marks]**

5. A sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = a_n - \frac{2n+1}{n^2(n+1)^2}, \quad \text{for all } n \geq 1.$$

(a) Use the method of mathematical induction to prove that $a_n = \dfrac{1}{n^2}$ for every positive integer $n$.

(b) Hence find

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2}.$$

(c) Give a reason why the series in (b) is convergent and find the sum to infinity.

(d) Use your answer in (b) to find

$$\sum_{k=2}^{n} \frac{2k-1}{k^2(k-1)^2}.$$

**[4, 4, 3, 4 marks]**

6. (a) (i) If $A, B, C$ and $D$ are any four events in a sample space $S$ such that $P[A \cap B \cap C] \neq 0$, show that

$$P[A \cap B \cap C \cap D] = P[A]\, P[B|A]\, P[C|A \cap B]\, P[D|A \cap B \cap C] .$$

(ii) A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives?

**[4, 5 marks]**

(b) We are given three coins: one has *heads* in both faces, the second has *tails* in both faces, and the third has a *head* in one face and a *tail* in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

**[6 marks]**

7. Let $P(x, y)$ be the point on an Argand diagram representing the complex number $u = x + iy$ and satisfying the equation

$$|u| = k|u + a|,$$

where $k$ is a non-negative real number and $a = \alpha + i\beta$ for some real numbers $\alpha$ and $\beta$.

(a) Assume that $k \neq 0$ and $k \neq 1$. Show that the locus of $P$ is a circle with centre at $\lambda a$ and radius $|a|\sqrt{\lambda(1+\lambda)}$, where $\lambda$ is the ratio $\dfrac{k^2}{1-k^2}$.

**[9 marks]**

(b) By making the substitution $u = z + b$ and making use of your answer for (a) describe the locus of the complex number $z$ if it satisfies the equation

$$|z + b| = k|z + c|,$$

where $b$ and $c$ are fixed complex numbers and $k$ is non-negative real number equal to neither 0 nor 1.

**[6 marks]**

8. (a) Given that $I_n = \displaystyle\int x^n \sqrt{1+x^2}\, \mathrm{d}x$, where $n$ is a non-negative integer, show that

(i) $I_0 = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right)\right) + C_0$. [Hint: Use the substitution $x = \sinh t$.]

(ii) $I_1 = \frac{1}{3}(1+x^2)\sqrt{1+x^2} + C_1$.

(iii) $I_n = \frac{1}{n+2}\left(x^{n-1}(1+x^2)\sqrt{1+x^2} - (n-1)I_{n-2}\right)$, for $n \geq 2$,

where $C_0$ and $C_1$ are the constants of integration.

**[3, 2, 6 marks]**

(b) The part of the curve $y = \frac{1}{2}x^2$ between $x = 0$ and $x = 2$ is rotated by $2\pi$ radians about the $x$-axis. Find the area of the surface of revolution so formed..

**[4 marks]**

9. (a) Given that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$,

(i) show that for all real values of $n$ and $x$,

$$(\sinh x + \cosh x)^n \equiv \sinh nx + \cosh nx\,;$$

(ii) prove that $f(x) \equiv \sinh x$ is an odd function and $g(x) \equiv \cosh x$ is an even function;

(iii) and hence deduce that

$$\int \frac{1}{\sqrt{\sinh x + \cosh x}}\,\mathrm{d}x = 2\left(\sinh\frac{x}{2} - \cosh\frac{x}{2}\right) + K\,,$$

where $K$ is the constant of integration.

**[2, 2, 3 marks]**

(b) The polar equation of the curve $C_1$ is given by $r = \sinh\theta$ for $0 \le \theta \le \pi$, and that of the curve $C_2$ is given by $r = -\sinh\theta$ for $-\pi \le \theta \le 0$.

(i) Sketch the two curves on the same set of axes.

(ii) Find the area enclosed by the two curves.

**[5, 3 marks]**

10. (a) Determine which of the following systems of equations is consistent and solve it.

(i)

$$\begin{aligned} x + y + z &= 1 \\ 2x + 4y - 3z &= 9 \\ 4x + 6y - z &= 10 \end{aligned}$$

(ii)

$$\begin{aligned} x - 2y + z &= 7 \\ 2x - y + 4z &= 17 \\ 3x - 2y + 2 &= 14 \end{aligned}$$

**[8 marks]**

(b) Using the adjoint method, find the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & -4 & -1 \\ 2 & -3 & 2 \\ -3 & 5 & -1 \end{pmatrix}.$$

Hence, or otherwise, find column vectors $\mathbf{v}$ and $\mathbf{w}$ such that

$$A(\mathbf{v}+\mathbf{w}) = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad A(\mathbf{v}-\mathbf{w}) = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}.$$

**[7 marks]**