MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2014

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	2nd SEPTEMBER 2014
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) If $y = e^{3x}\sin 4x$ show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 0\,.$$

[4 marks]

(b) A curve has equation $y = \dfrac{\ln x}{2x-3}$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Find the equation of the tangent to the curve when $x = 2$, giving your answer in the form $y = mx + c$.

[3, 3 marks]

2. (a) Find the following indefinite integrals:

$$\text{(i)}\ \int \frac{x+1}{(x-2)(x-3)}\,\mathrm{d}x \qquad \text{(ii)}\ \int (1+x)e^{-x}\,\mathrm{d}x$$

[3, 3 marks]

(b) Evaluate

$$\int_0^{\pi/4} \sin^4 x\,\mathrm{d}x\,.$$

[4 marks]

3. The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where a, b and c are constants such that $bc \neq 0$. Given that $\mathbf{P}^2 = (\mathbf{P}^2)^{\mathrm{T}}$ (where $\mathbf{A}^{\mathrm{T}}$ represents the transpose of a matrix **A**), show that:
(a) $a = -c$; and
(b) $|\mathbf{P}| \neq 0$.
It is given further that the matrix **P** represents a transformation mapping the line $y = 2x$ onto the line $y = x$.
(c) Evaluate the constants a, b and c.
(d) Find the image of the unit square under the transformation represented by **P**.

[2, 2, 3, 3 marks]

4. (a) Solve the following pair of simultaneous equations.

$$\log_2(x + y) = 2,$$
$$2\log_3 x = \log_3(10 - 3y).$$

[3 marks]

(b) The non-zero numbers a, k and b are three consecutive terms of an arithmetic series. Show that if the numbers a, 3 and b are the first three consecutive terms of a geometric progression then

$$a^2 - 2ka + 9 = 0.$$

Deduce that $|k| \geq 3$.
Given that $k = 5$, find the values of a and b for which the geometric series converges and find the sum to infinity in this case.

[3, 1, 3 marks]

5. (a) The function f is defined for $-1 \leq x \leq 0$ by

$$f(x) = \frac{1}{1 + x^2}.$$

Write down the range of f and find an expression for $f^{-1}(x)$ for x in the range of f.

[2, 3 marks]

(b) Let $f(x) = 3x + k$ and $g(x) = \dfrac{x-4}{3}$. For what value of k is $f(g(x)) = g(f(x))$?

[5 marks]

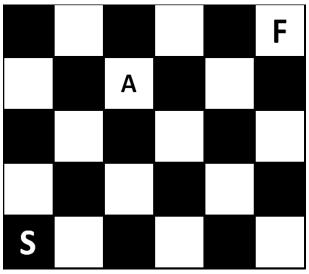
6. Solve the differential equation

$$y(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + x)e^{y^2},$$

given that $y = 0$ when $x = 0$.

[10 marks]

7. Consider the 5×6 checkerboard shown in the figure below.



Suppose that a coin is placed on the square labelled **S** and it has to be moved to the square labelled **F** such that, in each step of the path, the coin has to be moved to an adjacent box having a different colour.

(a) How many different paths from **S** to **F** are there such that the number of steps needed does not exceed 9?

[4 marks]

(b) Suppose that a path of 9 steps from **S** to **F** is randomly chosen. What is the probability that the coin has been on the square labelled **A**?

[6 marks]

8. Find the modulus and argument of z_1 and z_2 given that

$$z_1 = \frac{\sqrt{3} - \sqrt{3}\,i}{1+\sqrt{3}\,i} \qquad \text{and} \qquad z_2 = \cos\vartheta - \frac{1}{2(\cos\vartheta + i\sin\vartheta)}.$$

Show on an Argand diagram the complex numbers z_1 and z_2 given that $\vartheta = \pi/4$.

[3, 5, 2 marks]

9. Let A, B and C be points in the plane with coordinates $(7,1)$, $(3,-2)$ and $(4,5)$ respectively.

(a) Find the angle $\angle ABC$.

[4 marks]

(b) Find a and b if $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a unit vector in the direction of the line that bisects the angle $\angle ABC$. Give your answer to 4 decimal places.

[6 marks]

10. (a) If $x+3$ is a factor of $f(x)=2x^3+kx^2-4x-3$, find the value of k. Hence, express $f(x)$ as a product of three linear factors and resolve $\dfrac{1}{f(x)}$ into partial fractions.

[5 marks]

(b) Expand $\ln\dfrac{1+x}{1-x}$ as a series of ascending powers of x up to and including the term of x^5. Hence, or otherwise, show that

$$\ln 2 \approx \frac{2}{3}\left[1+\frac{1}{3}\left(\frac{1}{9}\right)+\frac{1}{5}\left(\frac{1}{9}\right)^2+\cdots\right].$$

[5 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2014

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	3rd SEPTEMBER 2014
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Consider the following system of equations.

$$\begin{aligned} x + 2y - 3z &= a, \\ kx - y + 2z &= b, \\ 3x - 8y + 13z &= c. \end{aligned}$$

   (a) By reducing to echelon form, show that when $k = 3$ the equations cannot have a unique solution and are consistent only if $2b = 3a + c$.

[6 marks]

   (b) Show that when $k = 1$ the three equations have a unique solution. Determine x, y and z in terms of a, b and c.

[9 marks]

2. The planes Π_1 and Π_2 have equations $x - y + 2z = 3$ and $2x + 3y - z = 1$ respectively.
   (a) Find a and b given that the point A with coordinates $(a, b, 0)$ lies on both Π_1 and Π_2. Find the vector equation of the line ℓ_1 where the planes Π_1 and Π_2 intersect.
   (b) Show that ℓ_1 is parallel to the plane Π_3 with equation $4x + y + 3z = -6$. Let ℓ_2 be the line perpendicular to Π_3 passing through A. Find the point B where ℓ_2 intersects Π_3.
   (c) Explain why the following system of equations does not have a solution:

$$\begin{aligned} x - y + 2z &= 3 \\ 2x + 3y - z &= 1 \\ 4x + y + 3z &= -6. \end{aligned}$$

   [Do NOT attempt to solve this system of equations.]

[6, 6, 3 marks]

3. Let the function f be given by

$$f(x) \equiv \frac{3x^2+4x-4}{1-x},$$

and consider the curve having equation $y_1 = f(x)$.

(a) Find the equations of all the linear asymptotes of the curve. Determine the coordinates of the stationary points on the curve and find their nature.

(b) Sketch the curve $y_1 = f(x)$, indicating clearly where the curve cuts the coordinate axes. State the range of values of $f(x)$.

[3, 3, 3, 2 marks]

Given that $y_2 = f(x+k)$ and $y_3 = f(x)+c$, find:

(c) the value of the constant k such that y_2 is not defined at $x = 0$;

(d) the range of values of the constant c such that y_3 does not take the value zero.

[2, 2 marks]

4. (a) Show that the equation $2\sinh x = \dfrac{5}{x}$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

[5 marks]

(b) (i) Use the series expansion of e^x to find the series expansion of $\cosh x$ up to and including the term in x^4. Estimate $\displaystyle\int_0^1 \cosh x \, \mathrm{d}x$ by integrating this series expansion.

(ii) Estimate $\displaystyle\int_0^1 \cosh x \, \mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to *four* decimal places.

(iii) Evaluate $\displaystyle\int_0^1 \cosh x \, \mathrm{d}x$ directly and compare your answer to those obtained in parts (i) and (ii).

[4, 3, 3 marks]

5. (a) Box 1 contains 6 blue discs and 4 red discs, while box 2 contains 3 blue discs, 3 red discs and 3 green discs. A disc is chosen at random from box 1 and placed in box 2, and then a disc is randomly selected from box 2. What is the probability that:

(i) the disc selected from box 2 is green?

(ii) the disc selected from box 2 is red?

(iii) the disc selected from box 1 was blue given that a blue disc was selected from box 2?

[2, 4, 5 marks]

(b) Given two events A and B neither of which is impossible, show that if $P[A|B] > P[A]$ then $P[B|A] > P[B]$.

[4 marks]

6. (a) Express the function

$$f(x) \equiv \frac{6 - 11x + 10x^2}{(1+x)(1-2x)^2}$$

in partial fractions. Hence, expand $f(x)$ as a series in ascending powers of x up to and including the term in x^3.

[9 marks]

(b) Use the method of mathematical induction to prove that

$$\text{if } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } A^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

for every positive integer n.

[6 marks]

7. (a) (i) Find the least value of $|z|$ if $|z-1| = |z+2+i|$.

(ii) Find the greatest value of $\arg z$ if $3|z-1| = 1$.

[5, 3 marks]

(b) Using Euler's identity $e^{i\vartheta} \equiv \cos\vartheta + i\sin\vartheta$, prove that if $z = \cos\vartheta + i\sin\vartheta$ then

$$z^n + z^{-n} = 2\cos n\vartheta,$$
$$z^n - z^{-n} = 2i\,\sin n\vartheta,$$

for any integer n. Show that

$$\cos^4\vartheta + \sin^4\vartheta \equiv \frac{1}{4}(\cos 4\vartheta + 3)$$

and hence find the general solution of the following equation.

$$2\cos^4\vartheta - 3\cos^2 2\vartheta + 2\sin^4\vartheta + 3\sin^2 2\vartheta = 0.$$

[4, 3 marks]

8. The curve C_1 has polar equation $r = \frac{\pi}{2}\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(a) Sketch the curve C_1.

[3 marks]

(b) Hence, or otherwise, sketch the curve C_2 having polar equation $r = (\pi - |\theta|)\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

[5 marks]

(c) Find the area enclosed by the curve C_2.

[7 marks]

9\. (a) (i) Given that $I_n = \int_0^{\frac{1}{2}\pi} t^n \sin t \; \mathrm{d}t$, show that for $n \geq 2$,

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}\,.$$

[4 marks]

(ii) A curve C in the xy-plane is defined parametrically in terms of t. It is given that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^4(1-\cos 2t) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = t^4 \sin 2t\,.$$

Find the length of the arc of C from the point where $t=0$ to the point where $t=\frac{1}{2}\pi$.

[6 marks]

(b) The area of an enclosed region is defined by $y \geq x^2+2$ and $y \leq 5x+2$. The region is rotated by 2π radians about the x-axis to form a solid of revolution. Show that the volume of this solid of revolution is 500π.

[5 marks]

10\. (a) (i) Show that, for $-1 < x < 1$,

$$\int \frac{2}{1-x^2}\,\mathrm{d}x = \ln\left(\frac{1+x}{1-x}\right) + C\,,$$

where C is an arbitrary constant.

(ii) Solve the differential equation

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{(1-x)(1-x^2)}{2+2x+x^2}\,,$$

for $-1 < x < 1$, given that $y = \ln 2$ when $x = 0$.

[2, 6 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 39e^{2x}\,,$$

given that $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

[7 marks]