MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2015

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	22nd MAY 2015
TIME:	16:00 to 19:00

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3y(y+2),$$
given that $y = 2$ when $x = \ln 2$, giving the solution in the form $y = \dfrac{2^p e^{6x}}{2^q - e^{6x}}$, where p, q are positive integers. Hence, find p and q.

[10 marks]

2. (a) Differentiate with respect to x:
   (i) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$;
   (ii) $\ln\left(\dfrac{1}{2x^2-1}\right)$.

[3, 2 marks]

(b) A curve has equation $x^3 - 3x^2y + y^3 + 1 = 0$. Use implicit differentiation to find the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the equation of the tangent to the curve at the point $(2, 3)$.

[5 marks]

3. The points A and B have position vectors $4\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $6\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, respectively.
   (a) Find the equation of the line ℓ_1 that passes through A and B.
   (b) Find the midpoint C of the line segment $\overrightarrow{AB}$.
   (c) Let D be the point with position vector $8\mathbf{i}-\mathbf{j}$. Show that $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{AB}$. Find the distance from D to the line ℓ_1.
   (d) Find the area of the triangle $\triangle ABD$.

[2, 2, 4, 2 marks]

4. Let $f(x) = x^3 - 6kx + k^3 + 8$, where k is a real number.
   (a) Show that $f(x)$ can be written in the form $(x+k+2)\cdot P(x)$, where $P(x)$ is a quadratic expression.

[5 marks]

   (b) Show that $2P(x)$ can be written as the sum of three expressions, each of which is a perfect square. Hence, or otherwise, solve the equation $f(x)=0$ for all values of k.

[5 marks]

5. Express $7\sin e^x - 24\cos e^x$ in the form $R\sin(e^x - \alpha)$ where $R > 0$ and α lies in the interval $[0, \pi/2]$.

[3 marks]

   Hence find:
   (a) the greatest and least values of

$$\frac{1}{7\sin e^x - 24\cos e^x + 30}.$$

[3 marks]

   (b) the values of x in the range $[0, \pi/2]$ for which

$$7\sin e^x - 24\cos e^x - 5 = 0.$$

[4 marks]

6. (a) By expressing $\dfrac{5+x}{(x-1)(5+x^2)}$ in partial fractions, find the value of p if

$$\int_2^3 \frac{5+x}{(x-1)(5+x^2)}\,\mathrm{d}x = \ln p.$$

[5 marks]

   (b) Use integration by parts twice to find

$$\int e^{3x}\cos 2x\,\mathrm{d}x.$$

[5 marks]

7. (a) A circle $\mathscr{C}$ has centre at the origin and radius 3. This circle is subject to a transformation T represented by the matrix

$$\mathbf{T} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Find the equation of the image of $\mathscr{C}$ under T and the area of the figure obtained.

[3, 2 marks]

(b) A group of students were given a mathematics test. From these, p% passed the test while f% failed. From past experience it is known that, if the students follow an intensive one-week course and a similar test is given at the end of the week, then the percentages change from $\mathbf{Z}$ to $\mathbf{XZ}$, where $\mathbf{Z} = \begin{pmatrix} p \\ f \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} 1 & 3/5 \\ 0 & 2/5 \end{pmatrix}$. However, if a similar test is given after a week of holidays, then $\mathbf{Z}$ changes to $\mathbf{YZ}$, where $\mathbf{Y} = \begin{pmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{pmatrix}$. Determine, in terms of p and f, the result obtained if a similar test is given at the end of:

(i) one week of holidays followed by one week of intensive course;

(ii) one week of intensive course followed by one week of holidays.

Which system from the two mentioned above gives the best results?

[2, 2, 1 marks]

8. (a) The function $f(x) = \dfrac{4}{x+2}$ is defined for all real $x \neq -2$ and the function $g(x) = x - 1$ is defined for all real x. If $h = g \circ f$ is the composite function of f and g, find an expression for:

(i) $h(x)$, giving its domain;

(ii) $h^{-1}(x)$, giving its domain, where h^{-1} is the inverse of h.

(b) Find the largest set D of values of x such that the function $f(x) = \dfrac{1}{\sqrt{3-x^2}}$ takes *real* values. Find the range of the function f defined on the domain D.

[2, 3, 3, 2 marks]

9. The positive integers can be split into three distinct arithmetic progressions, as shown below:

$$R_0 : 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ \ldots$$

$$R_1 : 1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ \ldots$$

$$R_2 : 2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20,\ \ldots$$

(a) Write down an expression for the value of the general term in each of the three progressions. Show that the integer 1706836 lies in R_1.

(b) Use the Binomial Theorem to show that if x is a term in R_1 or R_2 then x^6 is in R_1.

(c) Hence show that there are no positive integers x and y such that

$$x^6 + x + 3y = 1706836.$$

[3, 1, 3, 3 marks]

10. An energy agency conducted a survey on energy efficient measures adopted by households. The respondents were asked to state whether: they make use of energy efficient lighting; have an installed photovoltaic (PV) system; and/or have an installed solar water heater. From the 1000 respondents,

- 250 make use of energy efficient lighting and have a PV system,
- 350 make use of energy efficient lighting and have a solar water heater,
- 150 have a PV system and a solar water heater, and
- 50 adopt all the three energy efficient measures mentioned.

Find the probability that, upon choosing a respondent at random, he/she adopts:

(i) at most one of the energy efficient measures mentioned;

(ii) exactly two of the energy efficient measures mentioned.

Each respondent is given a lottery ticket for **each** of the energy efficient measure he/she adopts.

(iii) Given that a total of 1600 lottery tickets were distributed, determine the number of respondents who do not adopt any of the three energy efficient measures mentioned.

[4, 2, 4 marks]

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MATRICULATION CERTIFICATE EXAMINATION
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MAY 2015

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	23rd MAY 2015
TIME:	16:00 to 19:00

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) A curve that passes through the point $(1,2)$ is defined by the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(1+x^2-y).$$
Solve the differential equation giving your answer in the form $y = f(x)$.
[6 marks]

(b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x^2,$$
given that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.
[9 marks]

2. (a) (i) Let $I_n = \displaystyle\int_0^{\pi/2} \cos^n x \,\mathrm{d}x$. Show that
$$I_n = \frac{n-1}{n} I_{n-2}.$$
(ii) The region bounded by the curve $y = \sin x \cos^3 x$ and the x-axis between $x = 0$ and $x = \pi/2$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation. [Hint: use the identity $\sin^2 x + \cos^2 x = 1$.]
[5, 5 marks]

(b) The portion of the curve given by $y^2 = x$ between $y = 0$ and $y = 1$ is rotated through 2π radians about the x-axis. Find the area of the surface of revolution generated by rotation.
[5 marks]

3. (a) Show that the equation $3\sin x = x^2$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 2 as a first approximation. Do *two* iterations and give your working to *four* decimal places.
[Note that angles should be taken in radians throughout this question.]

[7 marks]

(b) (i) Evaluate $\displaystyle\int_0^1 \ln\left(1+\frac{x^2}{9}\right)\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to *four* decimal places.

(ii) Write down the series expansion of $\ln(1+\frac{x^2}{9})$ up to and including the term in x^4. Estimate the integral in part (i) by integrating this series expansion. Give your answer to *four* decimal places.

[4, 4 marks]

4. The curve $\mathscr{C}_1$ has equation $y = \dfrac{a+2x}{a-2x}$, where a is a negative constant.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ for all values $x \neq \frac{a}{2}$.

(b) Sketch the curve $\mathscr{C}_1$.

The curve $\mathscr{C}_2$ has equation $y = \left(\dfrac{a+2x}{a-2x}\right)^2$.

(c) Show that $\mathscr{C}_2$ has exactly one stationary point. Find its coordinates.

(d) Sketch the curve $\mathscr{C}_2$ on the same diagram drawn in (b).

(e) Use your sketch to show that it is possible to find values of the constant b such that the equation $b(a-2x)^3 - (a+2x)^2 = 0$ has 3 real and distinct roots. You do *not* need to give such a value of b, but you need to explain your reasoning.

[2, 4, 2, 4, 3 marks]

5. (a) (i) Show that $\ln(\sec x + \tan x)$ takes only real and positive values for real values of x between 0 and $\pi/2$ (not inclusive).

(ii) By starting from the definition of $\cosh x$, show that

$$\cosh(\ln(\sec x + \tan x)) \equiv \sec x$$

for $0 < x < \frac{\pi}{2}$.

(b) (i) Show that

$$(\sinh x + \cosh x)^4 \equiv \cosh 4x + \sinh 4x$$

and

$$(\sinh x - \cosh x)^4 \equiv \cosh 4x - \sinh 4x\,.$$

(ii) Hence, deduce that

$$\cosh 4x \equiv \cosh^4 x + 6\cosh^2 x \sinh^2 x + \sinh^4 x\,,$$

and obtain an expression for $\cosh 4x$ in terms of $\sinh x$ only.

[2, 3, 4, 6 marks]

6. (a) The lines ℓ_1 and ℓ_2 have vector equations $\mathbf{r} = 2\mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} + \mathbf{k} + \mu(\alpha\mathbf{i} - \mathbf{j} + \mathbf{k})$, respectively. Find α given that the two lines intersect at the point P. Find the position vector of P.
   (b) Find the angle between the lines ℓ_1 and ℓ_2.
   (c) Find the equation of the plane Π_1 that contains both ℓ_1 and ℓ_2.
   (d) Find the equation of the plane Π_2 that contains ℓ_1 and is perpendicular to Π_1.
   (e) Let Q be the point on the line ℓ_1 such that $\lambda = -2$, and R be the point on the line ℓ_2 such that $\mu = 1$. Find the area of the triangle $\triangle PQR$.

[3, 2, 3, 3, 4 marks]

7. (a) Find the seven roots of the equation

$$z^7 - 1 = 0, \qquad (\star)$$

   and plot them on an Argand diagram.
   (b) Let $w = \cos\dfrac{4\pi}{7} + i\sin\dfrac{4\pi}{7}$.
      (i) Show that

$$w - 1 = 2\sin\frac{2\pi}{7}\left(\cos\frac{11\pi}{14} + i\sin\frac{11\pi}{14}\right).$$

      (ii) Find the polar coordinates of the points of intersection of the loci $|z| = 1$ and $|z - w| = |z - 1|$.
      (iii) Show that if z is a root of equation $(\star)$ and the points $\{1, w, z\}$ are the vertices of an isosceles triangle then the area enclosed by this triangle is equal to either

$$\sin\frac{2\pi}{7} - \frac{1}{2}\sin\frac{4\pi}{7} \quad \text{or} \quad \sin\frac{4\pi}{7} + \frac{1}{2}\sin\frac{6\pi}{7}.$$

[4, 3, 3, 5 marks]

8. (a) The function f is defined by $f(x) = \ln\cos x$.
      (i) Find the first four derivatives of $f(x)$ and hence show that the first two non-zero terms of the Maclauren series for $f(x)$ are

$$-\frac{x^2}{2} - \frac{x^4}{12}.$$

      (ii) Use this series to find an approximate value for $\ln 2$, giving your answer to *two* decimal places.
   (b) The function f is defined by $f(x) = xe^{2x}$. Prove by the principle of mathematical induction that for every integer $n \geq 1$,

$$f^{(n)}(x) = (2^n x + n2^{n-1})e^{2x},$$

   where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

[4, 4, 7 marks]

9. Let the matrix $\mathbf{A}$ be given by

$$\mathbf{A} = \begin{pmatrix} 7 & 12 & k \\ 1 & 11 & -2 \\ 0 & 5 & -1 \end{pmatrix}.$$

(a) Solve the equation $\mathbf{Ax} = \mathbf{u}$ when $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $k = 0$.

[7 marks]

(b) Explain why the equation

$$\mathbf{Ax} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

always has at least one solution and that there is exactly one value of k for which the equation has more than one solution. Find this value of k and solve the equation for this value of k.

[8 marks]

10. (a) Out of a total of 30 different bottles, ten are made of glass and the rest are made of plastic. Eighteen bottles are chosen at random and put in one crate and the remaining bottles are put in another identical crate.

(i) In how many different ways can the bottles be placed in the crates?

(ii) In how many different ways can the bottles be placed in the crates such that one of the crates contains *exactly six* glass bottles?

(iii) Given that one crate was dropped and all the glass bottles in it got broken, find the probability that exactly six glass bottles got broken if:

(I) each crate has the same probability of being dropped;

(II) the probability that a crate is dropped is proportional to the number of bottles it contains.

[2, 2, 2, 3 marks]

(b) The events A and B are such that $P[A] = 0.15 + x$, $P[B] = 3x + 0.1$ and $P[A \cap B] = x$.

(i) Given that $P[A \cup B] = 0.55 + x$, find the values of x, $P[A]$ and $P[B]$.

Furthermore, the event C is such that $P[A \cap C] = 0$ and $P[A \cup B \cup C] = 1$.

(ii) Given that $P[C'] = 0.65$, where C' is the complement of the event C, find $P[B \cap C]$.

[4, 2 marks]