MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2015

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	1st SEPTEMBER 2015
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) If $y = e^{-1/x^2}$, show that

$$x^6 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2(3x^2 - 2)y = 0.$$

[4 marks]

(b) A curve is given parametrically by $x = \dfrac{t}{t^2+1}$ and $y = \dfrac{t^2}{t^2+1}$.

(i) Find the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(ii) Find the equation of the normal to the curve at the point where $t = \frac{1}{2}$.

[3, 3 marks]

2. Solve the differential equation

$$\cos^2 t \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x},$$

given that $x = 4$ when $t = \dfrac{\pi}{4}$. Find x when $t = \dfrac{\pi}{3}$, giving your answer correct to *three* significant figures.

[10 marks]

3. (a) Find A and B if

$$2\cos x + 3\sin x \equiv A(-\sin x + \cos x) + B(\cos x + \sin x).$$

Hence, find the indefinite integral

$$\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}\,\mathrm{d}x\,.$$

[5 marks]

(b) By using the substitution $x^2 = \dfrac{1}{u}$ evaluate

$$\int_1^2 \frac{\mathrm{d}x}{x^2\sqrt{5x^2-1}}\,.$$

[5 marks]

4. (a) Show that the parametric equations

$$x = 2 + 3\sin t \quad \text{and} \quad y = 4 + 3\cos\left(t + \frac{\pi}{2}\right)$$

represent the line having the point A with coordinates $(-1, 7)$ as one of its endpoints and the point B with coordinates $(5, 1)$ as its other endpoint.

(b) Determine the equation of the line which is perpendicular to the line AB and passes through the origin O.

(c) Find the equations of the two loci of the point P such that the area of triangle ABP is equal to a constant k.

(d) Hence, deduce the area of triangle OAB.

[3, 2, 4, 1 marks]

5. (a) Resolve into partial fractions.

$$\frac{x^2 - x - 4}{x^3 + 3x^2 - x - 3}\,.$$

[5 marks]

(b) A geometric progression has first term $\dfrac{1}{10}$ and common ratio $r < -1$. Given that every four consecutive terms a_k, a_{k+1}, a_{k+2} and a_{k+3} of this geometric progression satisfy the equation

$$3(a_{k+2} - a_k) = a_{k+1} - a_{k+3}\,,$$

show that $r^3 + 3r^2 - r - 3 = 0$. Hence, find the sum of the first eight terms of this progression.

[5 marks]

6. (a) The function $f(x) = e^{x^2}$ is defined for all real $x \geq 0$ and the function $g(x) = \dfrac{1}{x+3}$ is defined for all real $x \neq -3$. Let $h = g \circ f$ be the composite function of f and g.
   (i) Find an expression for $h(x)$, giving its domain;
   (ii) State the domain of $h^{-1}(x)$, where h^{-1} is the inverse of h;
   (iii) Find an expression for $h^{-1}(x)$.

[2, 1, 3 marks]

(b) Find the largest set of values of x such that the function $f(x) = \sqrt{\dfrac{8x-4}{x-3}}$ takes *real* values.

[4 marks]

7. For Sunday lunch, a restaurant offers its customers the opportunity to have a meal by choosing any three *different* dishes from a selection of eleven different dishes, ranging from appetizers, entrées, main course and desserts.
   (a) How many different meals can a customer avail of?

On a particular Sunday, the waiters report that they have served the following quantities of each dish:

$$17, 19, 5, 23, 36, 11, 4, 30, 22, 10, 14\,.$$

(b) Assuming that the waiters counted correctly, did all the customers have exactly three dishes? Give a brief explanation for your answer.

(c) If each dish is taken by the same number of people, and the number of customers on any one day is denoted by c, find an expression (in terms of c) for the quantity required of each dish. What can you deduce on the number of customers needed on any particular day for the above situation to be possible?

The eleven dishes are divided into two appetizers, three entrées, four main courses and two desserts. The management decided that a customer can have any *one* dish from each course up to a total of three dishes.

(d) How many different meals are now available for the customers?

[2, 2, 2, 4 marks]

8. (a) Find a and b given that

$$\frac{5+2i}{13+11i} = -\frac{1}{10}(a+ib), \quad \text{where } i = \sqrt{-1}\,.$$

If $a + ib$ is a root of the equation $x^2 + \alpha x + \beta = 0$, find the values of the real numbers α and β.

[5 marks]

(b) Given that $\cos\vartheta = \dfrac{5}{13}$ and that $\dfrac{3\pi}{2} \leq \vartheta \leq 2\pi$, show that $\sin 2\vartheta = -\dfrac{120}{169}$ and evaluate $\sin 3\vartheta$.

[5 marks]

9. The points A and B have position vectors $7\mathbf{i}-2\mathbf{j}+9\mathbf{k}$ and $-2\mathbf{j}+2\mathbf{k}$, respectively.
   (a) Find the equation of the line that passes through A and B.

   [3 marks]

   (b) Let C be the point with position vector $\alpha\mathbf{i}+\mathbf{k}$. Find the two possible values of α given that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BC}$.

   [3 marks]

   (c) For *one* of the values of α found in part (b), verify Pythagoras' Theorem for the triangle $\triangle ABC$

   [4 marks]

10. (a) Use the *method of completing the square* to show that the solutions of the following quadratic equation in x

$$x^2-2(3a^2+1)x+a^4-14a^2+1=0$$

   are given by $x=3a^2+1\pm 2a\sqrt{2a^2+5}$.
   Hence, solve the equation $x^2-26x-39=0$, giving your answer in surd form.

   [5 marks]

   (b) Solve the inequality

$$\sqrt{2x+3}\geq 2+\sqrt{x+4} \quad \text{for } x\geq \frac{-3}{2}.$$

   [5 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2015

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	2nd SEPTEMBER 2015
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix}$.
   (a) Show that $\mathbf{A}$ is non-singular and find $\mathbf{A}^{-1}$.
   [5 marks]
   (b) Find the equation of the image of the plane with Cartesian equation $x + y + z = 1$ under the transformation given by the matrix $\mathbf{A}$.
   [5 marks]
   (c) The graph of the cubic $y = ax^3+bx^2+cx+d$ passes through the points $(0,10), (1,7), (3,-11)$ and $(4,-14)$. Find the coefficients a, b, c, d.
   [5 marks]

2. The point A has position vector $10\mathbf{i} - \mathbf{j} + 8\mathbf{k}$.
   (a) Find the equation of the line ℓ_1 that passes through the point A and is perpendicular to the plane Π_1 with equation $3x - y + 2z = 5$. Find the position vector of the point B where the line ℓ_1 intersects Π_1.
   [4 marks]
   (b) Find the value of β such that the point C with position vector $\beta\mathbf{i} - 8\mathbf{j} - 2\beta\mathbf{k}$ lies in the plane Π_1. Find the equation of the plane Π_2 that contains A, B and C.
   [5 marks]
   (c) Find the angle $\angle ACB$, and the area of the triangle $\triangle ABC$.
   [6 marks]

3. The function f is given by $f(x) = \dfrac{x+7}{x^2+3x+8}$.
   (a) Determine the coordinates and nature of the stationary points and the equations of the asymptotes of the curve $y = f(x)$.

   [5 marks]

   (b) Sketch the curve $y = f(x)$.

   [4 marks]

   (c) Hence, or otherwise, sketch the curve of $y = -\dfrac{1}{f(x)}$ on a separate diagram.

   [4 marks]

   (d) Deduce the range of values of $f(x)$ and of $-\dfrac{1}{f(x)}$.

   [2 marks]

4. (a) Show that the equation $3\tan x + x = 0$ has a solution between 5 and 5.5. Use the Newton-Raphson method to find an approximate value of this solution, taking 5.25 as a first approximation. Do *two* iterations and give your working to *four* decimal places.
   [Note that angles should be taken in radians throughout this question.]

   [7 marks]

   (b) (i) Evaluate $\displaystyle\int_0^{\pi} \sin^2 x \, \mathrm{d}x$ by Simpson's Rule with an interval width of $h = \pi/4$. Give your answer in terms of π.
   (ii) Evaluate $\displaystyle\int_0^{\pi} \sin^2 x \, \mathrm{d}x$ by using the identity $2\sin^2 x = 1 - \cos 2x$ and compare your answer to that obtained in part (i).

   [4, 4 marks]

5. (a) Use De Moivre's Theorem to prove the identity

$$\cos 5\vartheta = 16\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta\,.$$

   Hence, or otherwise, find

$$\int \cos^5\vartheta \, \mathrm{d}\vartheta\,.$$

   [4, 3 marks]

   (b) Show on an Argand diagram the set of values of z for which

$$\left|\frac{z-3}{z+3}\right| \leq 2\,.$$

   What are the least values of $|z|$ and $|\arg z|$ when $\left|\dfrac{z-3}{z+3}\right| = 2$?

   [5, 1, 2 marks]

6. (a) Show that if A_1, A_2 and A_3 are three events, then:

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2|A_1]P[A_3|(A_1 \cap A_2)].$$

[4 marks]

(b) The manufacturer of a machine states that the probability that the machine first malfunctions in the first year is of 3%. If it has not malfunctioned in the first year, then the probability that it malfunctions in the second year increases to 5%. If it has not malfunctioned during the first two years, then there is a 10% chance that it malfunctions in the third year.

(i) Find the probability that the machine does *not* malfunction

(A) in the first year;

(B) in the second year, given that it does not malfunction in the first year;

(C) in the third year, given that it does not malfunction in the first two years.

(ii) Find the probability that the machine does *not* malfunction during the first three years.

According to the manufacturer, if the machine is given a proper service at the start of every year, then this should keep the probability that the machine malfunctions fixed at 3% each year.

(iii) What is the probability that the machine does *not* malfunction during the first three years if it is properly serviced every year?

[3, 4, 4 marks]

7. (a) The function f is defined by $f(x) = \sin^2 x$.

(i) Use the series of $\cos 2x$ to show that the first three non-zero terms of the Maclauren series for $f(x)$ are

$$x^2 - \frac{x^4}{3} + \frac{2x^6}{45}.$$

(ii) Hence find

$$\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}.$$

[4, 3 marks]

(b) Use the method of mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

for every positive integer n. Hence show that the sum of the first $(n+1)$ terms of the series

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \dots$$

is $\dfrac{(n+1)}{(2n+3)}$.

[8 marks]

8. Given that $f(\theta) = 6 + 2\sin 3\theta + 2\sin 5\theta$, show that:
   (a) $f(\pi - \theta) = f(\theta)$;
   (b) the equation $f(\theta) = 0$ has no real roots.

[3, 3 marks]

The curve $\mathscr{C}$ has polar equation $r = f(\theta)$.
   (c) Give a geometrical interpretation of the results proved in (a) and (b) for the curve $\mathscr{C}$.
   (d) Hence, by taking values of θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ in intervals of $\frac{\pi}{10}$ radians, sketch the curve $\mathscr{C}$ for $-\pi \le \theta \le \pi$.
   (e) Find the area enclosed by the curve $\mathscr{C}$.

[2, 4, 3 marks]

9. (a) Let $I_n = \displaystyle\int_0^1 x^n e^{-2x^2}\,\mathrm{d}x$. Show that

$$I_n = \frac{1}{4}\left((n-1)I_{n-2} - \frac{1}{e^2}\right) \quad \text{for } n \ge 2.$$

[5 marks]

   (b) Find I_4 given that $I_0 = 0.5982$. Give your answer to four decimal places.

[4 marks]

   (c) The region bounded by the curve $y = x^{5/2}e^{-x^2}$ and the x-axis between $x = 0$ and $x = 1$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[6 marks]

10. (a) Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \frac{x^3}{x^2+1},$$

given that $y = 1$ when $x = 1$, giving your answer in the form $y = f(x)$.

[6 marks]

   (b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x,$$

given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

[9 marks]