MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2016

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	30th APRIL 2016
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sqrt{y}\,\ln x}{x},$$
given that $y = 1$ when $x = e$.

[10 marks]

2. (a) Let $y = \ln\big(\sin(x^2)\big)$. Show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\cot(x^2) \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\sin(2x^2) - 4x^2}{\sin^2(x^2)}.$$

[4 marks]

(b) A curve is given by the parametric equations $x = \cos t$ and $y = 3\sin t$, where t takes values between 0 and 2π.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in terms of t.

(ii) Find the coordinates of the two points where the tangent to the curve has gradient $\sqrt{3}$. Show that the line joining these two points passes through the origin.

[6 marks]

3. The points A and B have position vectors $a\mathbf{i}+2\mathbf{k}$ and $-2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, respectively. The line ℓ_1 passes through A and is in the direction of $\mathbf{i}-2\mathbf{j}+\mathbf{k}$, while the line ℓ_2 passes through B and is in the direction of $3\mathbf{i}+\mathbf{j}+b\mathbf{k}$.
   (a) Find a and b given that ℓ_1 and ℓ_2 intersect each other and are perpendicular to each other.

   [3 marks]

   (b) Find the position vector of the point C where the two lines intersect.

   [1 mark]

   (c) Find the angle $\angle CAB$ and the area of the triangle $\triangle ABC$.

   [6 marks]

4. (a) Let z be a complex number. Show that $z+\dfrac{|z|^2}{z}$ is *real* and $z-\dfrac{|z|^2}{z}$ is *imaginary.*

   [4 marks]

   (b) Let $\alpha=|15-8i|$ and let $z^2=15-8i$. Evaluate α and find the two possible values of z. Hence, evaluate $z+\dfrac{\alpha}{z}$ for *one* value of z.

   [6 marks]

5. (a) Consider a circle with radius r and centre O.
   (i) Given that a chord AB subtends an angle θ at O, show that $AB=2r\sin\frac{\theta}{2}$.
   (ii) Let S denote the region obtained by subtracting the triangle AOB from the sector AOB (containing the triangle AOB). If the perimeter of S is equal to half the circumference of the circle, show that $\theta+2\sin\frac{\theta}{2}=\pi$.
   (iii) Verify that $\theta\approx 1.66$ (radians) and hence, estimate the area of S as a fraction of the area of the whole circle.

   [1, 1, 3 marks]

   (b) Find values of θ for $0\le\theta\le 360°$ which satisfy the equation

$$\operatorname{cosec}^2\theta=3+\cot\theta\,.$$

   [5 marks]

6. The points A and B have coordinates $(0,-q)$ and $(0,q)$, respectively, where q is a positive constant. A point P in the xy–plane moves such that $AP:PB=\mu$, where μ is a positive constant.
   (a) Determine the equation of the locus of P.
   (b) Show that the locus of P is a straight line if $\mu=1$.
   (c) Deduce that if $\mu\neq 1$, then P describes a circle. Find its centre C and its radius r.
   (d) Find the values of μ if the circle has radius $2q\sqrt{2}$.

   [3, 1, 3, 3 marks]

7. (a) By expressing $\dfrac{1}{x(x+1)(x+2)}$ in partial fractions, find the values of p and q if
$$\int_1^n \frac{1}{x(x+1)(x+2)}\,\mathrm{d}x = \frac{1}{2}\ln\frac{pn(n+2)}{q(n+1)^2}.$$
[4 marks]

(b) Evaluate
$$\int_0^{\ln 2} e^x(e^x+e^{-x})^2\,\mathrm{d}x.$$
[3 marks]

(c) Using the substitution $x=\frac{1}{2}\sin\theta$, or otherwise, evaluate
$$\int_0^{1/4}\sqrt{(1-4x^2)}\,\mathrm{d}x.$$
[3 marks]

8. The function $f(x)=\dfrac{2x+3}{x-4}$ is defined for all real $x\neq 4$ and the function $g(x)=\dfrac{x-1}{x+1}$ is defined for all real $x\neq -1$.
   (a) State the range for $f(x)$.
   (b) If $(c,0)$ is the x–intercept for the graph of $f(x)$, find the value of c.
   (c) Find $f^{-1}(x)$, where f^{-1} is the inverse of f.
   (d) Find the value of $g(f(3))$.
   (e) Find an expression for $g(f(x))$ and find the domain of the composite function $h=g\circ f$.

[1, 1, 3, 1, 4 marks]

9. (a) The equation $2x^2+3x+1=0$ has roots α and β. Find the equation whose roots are
$$\frac{\alpha}{\beta(1+\alpha^2+\beta^2)} \quad\text{and}\quad \frac{\beta}{\alpha(1+\alpha^2+\beta^2)}.$$
[5 marks]

(b) The first three terms of an arithmetic sequence have a sum of 24. The first, second and sixth terms of this arithmetic sequence are also consecutive terms of a geometric sequence. Find the first six terms of the arithmetic sequence.

[5 marks]

10. How many 5–digit codes are there between 00000 and 99999, both inclusive, having
   (a) the last digit 5;
   (b) the second digit 2 and the fourth digit 4;
   (c) at least one of the digits corresponding to its position (for example, the first digit 1, *or* the second digit 2, and so on).

[2, 2, 6 marks]

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MATRICULATION CERTIFICATE EXAMINATION
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MAY 2016

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	30th APRIL 2016
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Find the general solution of the differential equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k^2 y = 12\cos 3x\,,$$
where k is *real* and $|k| \neq 3$.
(b) Find the value of p if $p\,x\sin 3x$ is a particular integral of the differential equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9y = 12\cos 3x\,.$$
Hence, find the solution of this equation given that $y = \dfrac{4\pi}{3}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = \dfrac{\pi}{6}$.
[6, 9 marks]

2. (a) (i) Show that $1 + \cot^2 x = \operatorname{cosec}^2 x$.
(ii) Let $I_n = \displaystyle\int \operatorname{cosec}^n x \,\mathrm{d}x$. Show that, for $n \geq 2$,
$$(n-1)I_n = (n-2)I_{n-2} - \operatorname{cosec}^{n-2} x \cot x\,.$$
[Hint: Write $\operatorname{cosec}^n x = \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$, and use integration by parts.]
(iii) The region bounded by the curve $y = \operatorname{cosec}^2 x$ and the x–axis between $x = \pi/4$ and $x = 3\pi/4$ is rotated through 2π radians about the x–axis. Find the volume of the solid that is generated by this rotation.
[1, 5, 4 marks]
(b) A curve is given parametrically by $x = t - \sin t$ and $y = 1 - \cos t$. Find the length of the arc of the curve from the point where $t = 0$ to the point where $t = 2\pi$.
[5 marks]

3. [Note: Angles should be taken in radians throughout this question.]
   (a) Show that the equation $2\sin e^x = x$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

[7 marks]

   (b) (i) Evaluate $\int_0^1 \cos(x^2)\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to *four* decimal places.
   (ii) Write down the first *three* terms of the series expansion of $\cos(x^2)$. Estimate the integral in part (i) by integrating this series expansion. Give your answer to *four* decimal places.

[4, 4 marks]

4. The function f is given by $f(x) = \dfrac{(x-2)^2}{x^2+3x+2}$.
   (i) Find the coordinates of the points where the curve $y = f(x)$ cuts the coordinate axes.
   (ii) Determine the coordinates and the nature of the stationary points of the curve $y = f(x)$.
   (iii) Find the equations of the asymptotes of the curve $y = f(x)$.
   (iv) Sketch the curve $y = f(x)$.
   (v) Deduce the range of possible values of the constant a such that the equation $f(x) + a = 0$ does not have any real roots.

[2, 5, 3, 3, 2 marks]

5. The loci $\mathscr{C}_1$ and $\mathscr{C}_2$ have Cartesian equations $x = 4$ and $y^2 = 4x$, respectively.
   (i) Write the polar equation of $\mathscr{C}_1$.
   (ii) Show that the polar equation of $\mathscr{C}_2$ is given by $r = \dfrac{4\cos\theta}{\sin^2\theta}$, where $\sin\theta \neq 0$.
   (iii) Sketch $\mathscr{C}_1$ and $\mathscr{C}_2$ on the same diagram.
   (iv) Find the *polar coordinates* of the points of intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$.
   (v) Find the derivative with respect to θ of $\cot^3\theta$ and, using polar coordinates, determine the area enclosed between $\mathscr{C}_1$ and $\mathscr{C}_2$.

[1, 2, 4, 3, 5 marks]

6. The planes Π_1 and Π_2 have equations $x + y - 2z = 13$ and $2x - y + z = 3$, respectively.
   (a) Find the vector equation of the line ℓ_1 where the two planes intersect.
   (b) The point A has position vector $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. Let ℓ_2 and ℓ_3 be the lines through A that are perpendicular to the planes Π_1 and Π_2, respectively. Find the points B and C where these lines intersect the planes. Find the length BC.
   (c) Find the equation of the plane Π_3 that contains A and the line ℓ_1.

[4, 6, 5 marks]

7. The region in the Argand diagram given by the two inequalities

$$|z+40-9i| \geq 3|z-9i| \quad \text{and} \quad |z-5| \leq \frac{4}{5}|z-5-9i|$$

is the intersection of two circular regions.
(a) Find the centre and the radius of each circle.
(b) Find the points where they intersect the real axis.
(c) Sketch the circles and shade the given region on an Argand diagram.
(d) Show that the largest value of $|z|$ in this region is 17.

[8, 3, 2, 2 marks]

8. (a) The function f is defined by $f(x) = \dfrac{x^3 - x^2 + 1}{(x-1)^2(x^2+1)}$.
   (i) Express $f(x)$ in partial fractions.
   (ii) Find the first four terms when $f(x)$ is expanded in a series of ascending powers of x, stating the set of values of x for which the expansion is valid.

[3, 4 marks]

(b) Prove the following statements by the principle of mathematical induction.
   (i) $\sum_{m=1}^{n}(2^{m-1}-1) = 2^n - n - 1$, for all integers $n \geq 1$.
   (ii) $n+1 \leq 2^n \leq (n+1)!$, for all integers $n \geq 0$.

[4, 4 marks]

9. (i) Find the values of the constants a and b such that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -a & b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4$$

becomes the equation $x^2 - y^2 = 2$.
A linear transformation T is defined by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } \mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(ii) Find the matrix $\mathbf{T}^{-1}$.
(iii) Give the geometric interpretation of the transformation represented by $\mathbf{T}^{-1}$.
(iv) Hence, by writing $\begin{pmatrix} x \\ y \end{pmatrix}$ in terms of $\begin{pmatrix} X \\ Y \end{pmatrix}$, show that the equation $x^2 - y^2 = 2$ can be transformed into the equation $XY = 1$.
(v) By first sketching the graph of $XY = 1$, deduce and sketch the graph of $x^2 - y^2 = 2$.

[4, 2, 2, 4, 3 marks]

10. (a) An urn contains ten black balls and eight white balls. Every time a ball is randomly selected from the urn, its colour is noted and then it is returned in the urn together with an *additional* ball of the same colour.

A boy randomly selects three balls from the urn using the process described above.

(i) What should the first two selected balls be so that during the third selection, the probability of choosing a black ball is equal to the probability of choosing a white ball?

(ii) What is the probability that the three balls selected are of the same colour?

(iii) What is the probability that at least one black ball and one white ball are selected?

[2, 4, 3 marks]

(b) The interviewees applying for a job as cabin crew with an airline were asked to indicate whether they are fluent in English, French and Spanish. All the interviewees were fluent in at least one of the languages, 64 were fluent in exactly one language, and 36 were fluent in at least two languages. There were 63 fluent in English, 52 fluent in French and 25 fluent in Spanish.

(i) How many interviewees were there in total?

(ii) How many interviewees were fluent in the three languages?

[3, 3 marks]