MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2017

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	6th MAY 2017
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.

Each question carries 10 marks.

Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find P and Q if $e^{Px} = \dfrac{c(y-1)(x+Q)^2}{y+1}$, where c is a constant of integration, is a general solution of the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = x(y^2-1).$$

Find also the particular solution for which $y=2$ when $x=0$.

[8, 2 marks]

2. (a) Let $y = e^{-2\sqrt{x}}$. Show that

$$2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$$

[4 marks]

(b) A curve is given by the parametric equations

$$x = 4\cos t + 3\cos 3t \quad \text{and} \quad y = 4\sin t - 3\sin 3t\,,$$

where t takes values between 0 and 2π.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in terms of t.

(ii) Find the equation of the line that is tangent to the curve at the point where $t = \dfrac{2\pi}{3}$.

[2, 4 marks]

3. (a) The points A and B have position vectors $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $4\mathbf{i}-\mathbf{j}+\mathbf{k}$, respectively. Find the equation of the line ℓ_1 that passes through A and B.

[2 marks]

(b) The points C and D have position vectors $a\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $-3\mathbf{i}+3\mathbf{j}+b\mathbf{k}$, respectively. Let ℓ_2 be the line that passes through C and D. Find a and b given that ℓ_1 and ℓ_2 intersect at the midpoint of the line segment $\overrightarrow{AB}$.

[5 marks]

(c) Find the acute angle made by the lines ℓ_1 and ℓ_2.

[3 marks]

4. (a) Express $\dfrac{5x-12}{(x+2)(x^2-2x+3)}$ into partial fractions.

[4 marks]

(b) Solve the equation

$$\frac{1}{x+\sqrt{2}+\sqrt{3}}=\frac{1}{x}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}},$$

giving your answer in surd form.

[6 marks]

5. (a) (i) Find the locus of a point P whose distance from the point A with coordinates $(1,2)$ is twice its distance from the origin.

(ii) A point B has coordinates $(0,1)$. Find the locus of the midpoint of BP.

(iii) Show that the points of intersection of the two loci lie on the line with equation $2x+10y=7$.

[2, 3, 2 marks]

(b) A group of 11 sopranos, 8 altos, 5 tenors and 3 basses were chosen after an audition to form part of a choir. However, in the choir there are places for only 6 sopranos, 5 altos, 4 tenors and 3 basses. Moreover, two of the sopranos are archrivals and they will not accept to be in the choir together. In how many different ways can the choir be formed.

[3 marks]

6. (a) Find the coefficient of x^9 in the expansion of $\left(1-3x^2\right)\left(x^3-\frac{2}{x}\right)^7$.

[5 marks]

(b) Show that

$$\frac{1}{\log_2 x}+\frac{1}{\log_3 x}+\frac{1}{\log_4 x}+\cdots+\frac{1}{\log_{100} x}=\frac{1}{\log_{100!} x}.$$

[2 marks]

(c) Find the sum of the positive terms of the following sequence.

$$87, 84, 81, 78, \ldots.$$

[3 marks]

7. (a) By expressing $\dfrac{x+7}{x^2(x+2)}$ in partial fractions, find the values of p and q if

$$\int_1^2 \frac{x+7}{x^2(x+2)}\,\mathrm{d}x = p\left(\ln\frac{2}{3}+q\right).$$

[6 marks]

(b) Using the substitution $u=\cos\theta$, or otherwise, find

$$\int \cos^4\theta\,\sin^3\theta\;\mathrm{d}\theta\,.$$

[4 marks]

8. (a) Consider the function $f(x) = -2x^2-4x$ which is defined for all real $x\le -1$.
   (i) Express $f(x)$ in the form $a(x-h)^2+k$ and state the values of a, h and k. Find the range of $f(x)$.
   (ii) Find an expression for the inverse function $f^{-1}(x)$ and state its domain and range.

[2, 3 marks]

(b) The function $g(x)=e^x+1$ is defined for all real x and the function $h(x)=(x-1)\ln(x-1)$ is defined for all real $x>1$.
   (i) Find an expression for the composite function $(h\circ g)(x)$ and find its domain.
   (ii) Find the **two** real values of x that satisfy $(h\circ g)(x)=2x$.

[3, 2 marks]

9. (a) Write down the matrix **R** representing a reflection in the line $y=x\tan\beta$, where $0<\beta<\frac{\pi}{2}$.

[1 marks]

(b) Show that $\mathbf{R}^2=\mathbf{I}$, where **I** is the identity matrix. Deduce the inverse of the matrix **R**.

[3 marks]

(c) Write down the matrix **A** representing an anticlockwise rotation through an angle α about the origin, where $0<\alpha<\pi$.

[1 marks]

(d) Show that the matrix representing the composite transformation obtained by first applying **R** and then applying **A** is given by

$$\begin{pmatrix}\cos(\alpha+2\beta) & \sin(\alpha+2\beta)\\ \sin(\alpha+2\beta) & -\cos(\alpha+2\beta)\end{pmatrix}.$$

[3 marks]

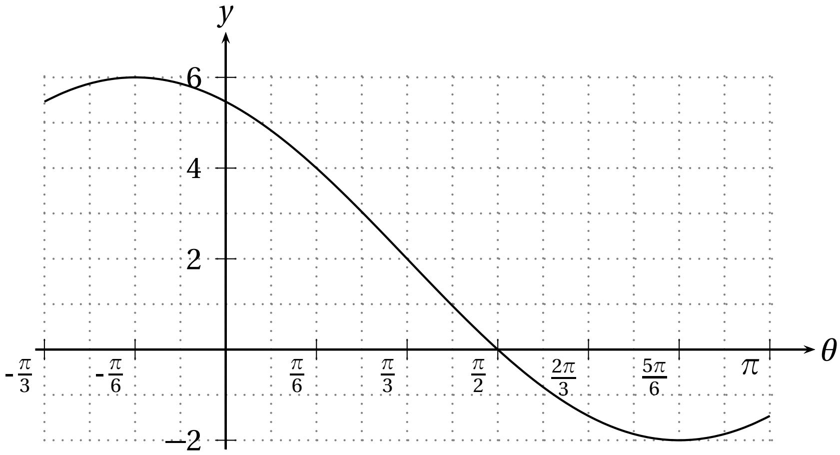
(e) The matrix **B** represents a reflection in the y-axis. Given that the composite transformation represented by the matrix product **BAR** is equivalent to the identity transformation, find a relationship between α and β.

[2 marks]

10. (a) Let $z = 5 + 12i$ and $w = x - 4i$, where $i = \sqrt{-1}$ and $x > 0$.
   (i) Find the value of x given that $|wz| = 65$.
   (ii) For this value of x, express $\frac{w}{\sqrt{z}}$ in the form $a + bi$, where a and b are rational numbers.

[1, 4 marks]

(b) The diagram shows part of the graph $y = p + r\sin(\theta + s)$, where p, r and s are constants and $0 \le s \le \pi$.



   (i) Find the values of p, r and s.
   (ii) Give the general solution of the equation $y = 2\sqrt{3}\cos\theta$.

[3, 2 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2017

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	6^{th} MAY 2017
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the differential equation
$$2\cos x\frac{\mathrm{d}y}{\mathrm{d}x}+4y\sin x=\sin 2x\,,$$
given that $y=0$ when $x=\dfrac{\pi}{3}$.

[8 marks]

(b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+y=e^{-x}\,,$$
given that $y=1$ when $x=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=2$.
[When finding the particular integral, use Px^2e^{-x} as a trial solution.]

[7 marks]

2. (a) Show that the equation $e^{-x^2}+2=x^2$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1.5 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[7 marks]

(b) (i) Evaluate $\displaystyle\int_0^{0.6} e^{-x^2}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=0.3$. Give your answer to **four** decimal places.

(ii) Write down the series expansion of e^{-x^2} up to, and including, the term in x^6. Estimate the integral in part (i) by integrating this series expansion. Give your answer to **four** decimal places.

[4, 4 marks]

3. (a) (i) Use integration by parts to evaluate the indefinite integral $\int x\cos x \, dx$.

(ii) Let $I_n = \int_0^{\frac{\pi}{2}} x\cos^n x \, dx$. By writing $x\cos^n x = (x\cos x)\cos^{n-1} x$, and using the result of part (i), show that, for $n \geq 2$,

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} x\sin^2 x\cos^{n-2} x \, dx - \frac{1}{n}.$$

Deduce that $n^2 I_n = n(n-1)I_{n-2} - 1$.

(iii) The region bounded by the curve $y = x^{\frac{1}{2}}\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[1, 5, 4 marks]

(b) A curve is given parametrically by $x = t^3$ and $y = 3t^2$. Find the length of the arc of the curve from the point where $t = 0$ to the point where $t = 4$.

[5 marks]

4. The function f is given by $f(x) = \dfrac{2(x+4)}{x-1}$.

(a) Determine the equations of the asymptotes of the curve $y = f(x)$.

[2 marks]

(b) Sketch the curve of $y = f(x)$.

[5 marks]

(c) Hence, or otherwise, sketch the curve of $y = |f(x)|$ on the same diagram used in (b).

[3 marks]

(d) Deduce the values of x such that $|f(x)| \geq 2$.

[2 marks]

(e) Using a graphical technique, explain why the equation $|f(x)| = 2x + 8$ has three roots, and deduce the value of these roots.

[3 marks]

5. [Note: Throughout this question, r and θ are polar coordinates with $r \geq 0$ and $-\pi < \theta \leq \pi$.]
The curve $\mathscr{C}_1$ has Cartesian equation $(x^2 + y^2 + y)^2 = 9(x^2 + y^2)$.

(a) Find the polar equation of $\mathscr{C}_1$ and sketch the curve $\mathscr{C}_1$.

[5 marks]

The curve $\mathscr{C}_2$ has polar equation $r = 2(1 - \sin\theta)$.

(b) Find the tangent(s) to the curve $\mathscr{C}_2$ at the pole.

[2 marks]

(c) Sketch the curve $\mathscr{C}_2$ on the same diagram used in (a).

[2 marks]

(d) Find the area enclosed between the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$.

[6 marks]

6. The points A, B and C have position vectors $2\mathbf{i}-\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}$ and $2\mathbf{i}+3\mathbf{j}+11\mathbf{k}$, respectively.
   (a) Find the equation of the plane Π_1 that passes through C and is perpendicular to $\overrightarrow{AB}$.
   [3 marks]
   (b) The line ℓ_1 passes through A and B. Find the position vector of the point D where ℓ_1 intersects the plane Π_1. Evaluate the dot product $\overrightarrow{AB}\cdot\overrightarrow{CD}$, and comment on your answer.
   [4 marks]
   (c) Find the plane Π_2 that passes though A and B and is perpendicular to $\overrightarrow{CD}$. Find the vector equation of the line ℓ_2 where the planes Π_1 and Π_2 intersect.
   [4 marks]
   (d) Find the area of the triangle $\triangle ABC$ by
      (i) evaluating the vector product $\overrightarrow{AB}\wedge\overrightarrow{AC}$, and
      (ii) finding the lengths of $\overrightarrow{AB}$ and $\overrightarrow{CD}$.
   [2, 2 marks]

7. (a) Prove that
$$(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi),$$
   where $i=\sqrt{-1}$. Use mathematical induction to show that, for every positive integer n,
$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta\,.$$
   [1, 2 marks]
   (b) Solve for x the equation $x+x^{-1}=2\cos\alpha$, giving your answer in terms of α. Hence, show that $x^n+x^{-n}=2\cos n\alpha$.
   [3, 2 marks]
   (c) Find the zeros of the equation $z^7-1=0$, giving your answers in the form $r(\cos\theta+i\sin\theta)$ where $r>0$ and $0\leqslant\theta<2\pi$. Hence, or otherwise, show that
$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}\,.$$
   [4, 3 marks]

8. (a) By direct application of the Maclauren expansion formula, find the Maclauren expansion, as far as the term in x^4, of
$$f(x)=e^{2x}\cos 3x\,.$$
   [8 marks]
   (b) Prove the following statement by the principle of mathematical induction:
$$\frac{1}{(4\times 1^2)-1}+\frac{1}{(4\times 2^2)-1}+\frac{1}{(4\times 3^2)-1}+\cdots+\frac{1}{(4\times n^2)-1}=\frac{n}{2n+1},$$
   for all integers $n\geq 1$.
   [7 marks]

9. (a) Prove that

$$\begin{vmatrix} (a+b)^2 & 1 & 1 \\ a^2 & (1+b)^2 & a^2 \\ b^2 & b^2 & (1+a)^2 \end{vmatrix} = 2ab(1+a+b)^3.$$

[7 marks]

(b) Deduce that the equations

$$\begin{array}{ccccccc} (2+k)^2x & + & y & + & z & = & 0 \\ 4x & + & (1+k)^2y & + & 4z & = & 0 \\ k^2x & + & k^2y & + & 9z & = & 0 \end{array}$$

have a non-trivial solution if and only if $k=0$ or $k=-3$.
Find the solution set of the equations for the cases $k=0$ and $k=-3$.

[3, 3, 2 marks]

10. An experiment consists of tossing a fair die having six faces numbered from 1 to 6 and simultaneously tossing a fair coin. If the coin shows heads, the score is equal to double the number shown on the die, whereas if the coin shows tails, the score is equal to the number shown on the die.

(a) Find the probability that the score obtained in the experiment is an even number.

[4 marks]

The experiment is repeated five times. What is the probability that

(b) the score is an even number in all the five experiments;

[3 marks]

(c) the score is an even number in exactly one of the five experiments.

[3 marks]

The total final score is obtained by adding the scores of the five experiments.

(d) What is the probability that the total final score is even.

[5 marks]