MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2017

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	4th SEPTEMBER 2017
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.

Each question carries 10 marks.

Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Let $y = x^3 \sin(\ln x)$. Show that

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5x \frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 0.$$

[5 marks]

(b) A curve has equation $y^3 - 3xy^2 + 2x^2 y = 3$. Find the equation of the tangent to the curve at the point $(2, 1)$.

[5 marks]

2. Solve the differential equation

$$\operatorname{cosec}^3 x \frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = 0$.

[10 marks]

3. (a) Evaluate the integral

$$\int_0^1 e^{5x}\left(\frac{e^{2x}}{7}+\frac{3}{e^{3x}}\right)\mathrm{d}x\,.$$

[4 marks]

(b) (i) Show that for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x+\tan x)=\sec x\,.$$

(ii) Hence, or otherwise, find

$$\int \cos x\ln(\cos x)\,\mathrm{d}x\,,\quad \text{with } -\frac{1}{2}\pi < x < \frac{1}{2}\pi\,.$$

[3, 3 marks]

4. The function f is given by $f(x)=-\frac{1}{8}\left(x^4-6x^3+4x^2+24x-32\right)$.

(a) Show that the curve $y=f(x)$ intersects the x-axis at $x=-2$ and at $x=4$.

[2 marks]

(b) Find the coordinates of the stationary points of the curve $y=f(x)$.

[4 marks]

(c) Hence sketch the curves of the following equations on the same set of axes:

(i) $y=f(x)$;

(ii) $y=f(x+1)$.

[4 marks]

5. (a) Express $\cos\theta-\sqrt{3}\sin\theta$ in the form $R\cos(\theta+\alpha)$, where R is a positive number and α is an angle measured in radians between 0 and $\frac{\pi}{2}$.
Hence, sketch the graph of $y=\cos\theta-\sqrt{3}\sin\theta$, given that $0\leqslant\theta\leqslant 2\pi$, showing clearly the intercepts of the graph with the lines $y=0$, $y=\pm1$ and $y=\pm2$.

[7 marks]

(b) Find the sum of the infinite geometric series

$$1+\frac{\sec\theta}{1-\sqrt{3}\tan\theta}+\frac{\sec^2\theta}{(1-\sqrt{3}\tan\theta)^2}+\frac{\sec^3\theta}{(1-\sqrt{3}\tan\theta)^3}+\cdots\,,$$

giving your answer in terms of θ. For what values of θ in the interval $[0,2\pi]$ is your answer valid?

[3 marks]

6. The function $g(x) = 2x - 1$ is defined for all real x and the function $h(x) = \dfrac{1}{x+3}$ is defined for all real $x \neq -3$.
   (a) Find $g^{-1}(x)$ and show that $g^{-1}(g(x)) = x$.

   [2 marks]

   (b) Find an expression for $g(h(x))$ and find the domain of the composite function $g \circ h$.

   [3 marks]

   (c) Sketch a complete and accurate graph of $g \circ h$. Clearly label the x- and y-intercepts, and any asymptotes in your sketch.

   [4 marks]

   (d) State the range of $g \circ h$.

   [1 marks]

7. Let $\mathbf{C}$ be the matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 9 & 3 \end{pmatrix}$.

   (a) Show that $\mathbf{C}^3 = 13\mathbf{C}^2 - 144\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

   [5 marks]

   (b) Given that $\mathbf{C}^{-1}$ exists, **without finding** $\mathbf{C}^{-1}$, show that this equation can be written as $\mathbf{C}^{-1} = \frac{1}{144}\mathbf{C}(13\mathbf{I} - \mathbf{C})$.

   [2 marks]

   (c) Hence deduce $\mathbf{C}^{-1}$.

   [3 marks]

8. (a) If $\log_r p = q$ and $\log_q r = p$, prove that $\log_q p = pq$.

   [3 marks]

   (b) Consider the complex number

$$z = \frac{\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^2 \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - i\sin\frac{\pi}{24}\right)^4}.$$

   (i) Find the modulus and the argument of z.
   (ii) Simplify $(1+2z)(2+z^2)$ expressing your answer in the form $a + bi$, leaving a and b in surd form. Show that $(1+2z)(2+z^2) + 3\overline{z} = 0$.

   [2, 5 marks]

9. The lines ℓ_1, ℓ_2 and ℓ_3 have vector equations

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} - \mathbf{k}),$$
$$\mathbf{r} = 3\mathbf{i} - 11\mathbf{j} - 4\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 3\mathbf{k}),$$
$$\text{and } \mathbf{r} = 6\mathbf{i} - 19\mathbf{j} + 5\mathbf{k} + \nu(\mathbf{i} - 10\mathbf{j} + 3\mathbf{k}),$$

respectively.

(a) Show that ℓ_3 is perpendicular to both ℓ_1 and ℓ_2.

[4 marks]

(b) Show that ℓ_3 intersects both ℓ_1 and ℓ_2, and find the points of intersection.

[4 marks]

(c) Find the distance between the two points of intersection found in (b).

[2 marks]

10. Let $p(x) = x^3 + ax^2 + bx + 6$, where a and b are constant real numbers.

(a) Show that if the equation $p(x) = 0$ has three real distinct roots, then $a^2 > 3b$.

[4 marks]

(b) It is given that $(x-2)$ is a factor of $p(x)$, and that when divided by $x-1$, the polynomial $p(x)$ leaves a remainder 4. Find the values of a and b, and factorise $p(x)$ completely.

[6 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2017

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	5th SEPTEMBER 2017
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Let $\mathbf{A} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & \cos\theta & 0 \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

   (a) (i) Show that $|\mathbf{A}| = \cos^3\theta + \sin^3\theta$ and find the value of θ for which $\mathbf{A}$ is singular.
   (ii) Solve for $\mathbf{x}$ the equation $\mathbf{Ax} = \mathbf{0}$ for this value of θ.

   [4, 4 marks]

   (b) Solve for $\mathbf{x}$ the equation $\mathbf{Ax} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, when $\theta = \frac{\pi}{3}$.

   [7 marks]

2. (a) Sketch the graphs of $y = -x(x+3)^2$ and of $y^2 = -x(x+3)^2$ on the same set of axes.

   [5 marks]

   (b) The function f is given by

$$f(x) = \frac{(2x+3)(x-1)}{(2x-1)(x+1)}.$$

   (i) Find the range of values of f.
   (ii) Find the equation of the horizontal asymptote of the curve $y = f(x)$.
   (iii) Deduce the only stationary value of $y = f(x)$, giving reasons for your answer.

   [5, 2, 3 marks]

3. The planes Π_1 and Π_2 have equations $2x + y - 3z = 1$ and $4x - y + z = 3$, respectively.
   (a) Find the angle between the planes Π_1 and Π_2, and the vector equation of the line ℓ_1 where the planes intersect.

   [7 marks]

   (b) Find the equation of the plane Π_3 that contains the line ℓ_1 and the point A with position vector $2\mathbf{i} - 11\mathbf{j} - 6\mathbf{k}$.

   [4 marks]

   (c) Find the vector equation of the line ℓ_2 that lies in the plane Π_3, is perpendicular to ℓ_1 and passes through A.

   [4 marks]

4. [Note: Angles should be taken in radians throughout this question.]
   (a) Show that the equation $x \sin x = 3 \cot x$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

   [7 marks]

   (b) (i) Express the length of the portion of the curve $y = \cos x$ between $x = 0$ and $x = \pi$ as an integral.
   (ii) Use Simpson's Rule to estimate this integral. Use an interval width of $h = \dfrac{\pi}{6}$ and give your answer to **four** decimal places.

   [3, 5 marks]

5. (a) Write down Euler's identity. Use the trigonometric factor formulae to show that

$$e^{i\theta} + e^{i\phi} = 2\cos\frac{\theta - \phi}{2}\, e^{\frac{\theta+\phi}{2} i},$$

   where $i = \sqrt{-1}$.
   Hence, show that the solutions of the equation

$$\left(\frac{z+1}{z-1}\right)^5 = 1$$

   have the form $z = -i \cot \frac{k\pi}{5}$, where $k \in \{1, 2, 3, 4\}$.

   [9 marks]

   (b) $P(x, y)$ is the point on an Argand diagram representing $z = x + iy$, where x and y are real numbers. Given that $|z + 2| = 3|z - 2 - 4i|$, find the maximum value of $|z|$.

   [6 marks]

6. The curve $\mathscr{C}_1$ has polar equation $r = \frac{1}{5}(4 + \sin 5\theta)$, where $0 \le \theta \le 2\pi$.
   (a) Sketch the curve $\mathscr{C}_1$. **[3 marks]**
   The circle $\mathscr{C}_2$ has Cartesian equation $x^2 + y^2 = 1$.
   (b) Find the polar equation of $\mathscr{C}_2$ and sketch it on the same diagram used in (a). **[2 marks]**
   (c) Find the polar coordinates of the points of intersection of the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$. **[4 marks]**
   (d) Find the area enclosed between the two curves. **[6 marks]**

7. In a survey about the use of mobile phones by passengers during any particular bus journey, it was found that of those passengers using their mobile phone, 60% use it to access the Internet, 45% to make/receive a call, and 35% to send an SMS. There are 5% who use their mobile phone for all the three purposes during one journey.
   (a) What percentage of the passengers who use their mobile phone do so for only one of the above mentioned purposes? **[5 marks]**
   The survey also concluded that 22% of the passengers on a bus do **not** use their mobile phones.
   (b) If a bus is chosen at random, what is the probability that a randomly chosen passenger uses his/her mobile phone for only one of the above mentioned purposes? **[2 marks]**
   Three buses are chosen at random and a passenger is randomly chosen from each bus.
   (c) What is the probability that the three chosen passengers do not use their mobile phones? **[3 marks]**
   (d) What is the probability that exactly two of the chosen passengers use their mobile phone for only one of the three different purposes mentioned above? **[5 marks]**

8. (a) (i) By direct application of the Maclauren expansion formula, find the Maclauren expansion, as far as the term in x^4, of $f(x) = \ln(1 + x)$.
   (ii) Hence, or otherwise, find the first **four** non-zero terms of the Maclauren expansion of
$$g(x) = \frac{\ln(1+2x)}{\sqrt{1+x}}.$$
[3, 5 marks]
   (b) A sequence $a_1, a_2, a_3, \dots$ is such that $a_1 = 0$, $a_2 = -6$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \ge 3$. Use the method of mathematical induction to prove that
$$a_n = 3 \times 2^n - 2 \times 3^n$$
   for every positive integer n. **[7 marks]**

9. (a) Find the general solution of the differential equation

$$3x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 1 + \ln x\,, \qquad \text{for } x > 0\,,$$

giving y explicitly in terms of x. Find also the particular solution for which $y = -2$ when $x = 1$.

[8 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 10\sin x\,,$$

given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

[7 marks]

10. (a) (i) Let $I_n = \int_0^{\pi/2} \sin^n x \; \mathrm{d}x$. Show that, for $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}\,.$$

(ii) The region bounded by the curve $y = \sin^{7/2} x$ and the x–axis between $x = 0$ and $x = \pi/2$ is rotated through 2π radians about the x–axis. Find the volume of the solid that is generated by this rotation.

[4, 4 marks]

(b) A function y is defined by $x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$.

(i) Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$, and show that $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{(x+1)^2}{4x}$.

(ii) The part of the curve of this function y between $x = 0$ and $x = 3$ is rotated by 2π radians about the x–axis. Find the area of the surface of revolution so formed.

[3, 4 marks]