MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2018 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	3rd September 2018
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1+e^{-x})(y^2-1).$$

Find also the particular solution for which $y=0$ when $x=0$.

[8, 2 marks]

2. (a) Let $y = x\ln x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$(x^2+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1\,.$$

[4 marks]

(b) A curve has equation $x^2y^2 - xy + x^2 = 3$. Find the equation of the tangent to the curve at the point $(1,2)$.

[6 marks]

3. (a) The points A and B have position vectors $\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$, respectively. Find the vector equation of the line ℓ_1 that passes through A and B.

[2 marks]

(b) The line ℓ_2 with vector equation $\mathbf{r}=6\mathbf{i}+6\mathbf{j}+\mathbf{k}+\mu(4\mathbf{i}+3\mathbf{j}+2\mathbf{k})$ intersects ℓ_1. Find the position vector of the point of intersection C of ℓ_1 and ℓ_2.

[5 marks]

(c) The vector $\mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k}$ is perpendicular to both lines. Find α and β.

[3 marks]

4. When the function $f(x)=2x^3+ax^2+bx+4$ is divided by $x-1$ the remainder is 3, and $x-2$ is a factor of $f(x)$.

(a) Show that $a=-7$ and $b=4$. Factorise $f(x)$ completely.

[5 marks]

(b) Express the following rational function in partial fractions:

$$\frac{7(6-x^2)+18x}{f(x)}.$$

[5 marks]

5. A point P is equidistant from the points A and B having coordinates $(4,5)$ and $(5,-2)$, respectively.

(a) Find the locus ℓ_1 of P.

[2 marks]

(b) Determine the **two** possible sets of coordinates of the point P such that the lines AP and BP are perpendicular.

[3 marks]

(c) Find the locus ℓ_2 of the line through A which is perpendicular to ℓ_1.

[2 marks]

(d) Find the coordinates of the point C of intersection of ℓ_1 and ℓ_2.

[2 marks]

(e) Hence determine the equation of the circle with centre C which touches the x-axis.

[1 marks]

6. The fourth term of an arithmetic progression is 3, the last term is 30, and the sum is equal to $\frac{627}{2}$. Find the first term, the common difference and the number of terms of this progression.

[10 marks]

7. (a) Use integration by parts to find

$$\int x^3 \ln 5x \, \mathrm{d}x\,.$$

[4 marks]

(b) Using the substitution $u = 4 - \sqrt{x}$, or otherwise, show that

$$\int_0^9 \sqrt{4-\sqrt{x}}\, \mathrm{d}x = \frac{188}{15}\,.$$

[6 marks]

8. (a) Consider the function $f(x) = \dfrac{1}{x+3} - 5$ which is defined for all real $x \neq -3$. Find an expression for the inverse function $f^{-1}(x)$ and state its domain and range.

[4 marks]

(b) The function $g(x) = x^2 - 7x + 3$ is defined for all real x and the function $h(x) = \sqrt{x-5}$ is defined for all real $x \geq 5$. Find an expression for the composite functions $(h \circ g)(x)$ and $(g \circ h)(x)$ and find their domains (as composite functions).

[6 marks]

9. A delivery van has enough time to make only six out of the ten deliveries that are scheduled for today.

(a) In how many different ways can the driver choose the six deliveries to make (noting that no restrictions are being made)?

[2 marks]

(b) What is the probability that a particular shop receives its delivery today?

[3 marks]

Two of the ten scheduled deliveries are to be made to shops in the same locality A, whereas all the other eight deliveries are to shops in another eight different localities (different also from locality A). The policy of the driver is that deliveries to shops in the same locality are done on the same day.

(c) In how many different ways can the driver choose the six deliveries to make, keeping in mind the above mentioned policy?

[5 marks]

10. (a) Find the range of values of x such that

$$\frac{2x-1}{2x-3} > \frac{x-4}{x-2}.$$

[5 marks]

(b) Prove that if $a+b+c=0$, then

$$a^3+b^3+c^3=3abc.$$

Hence, or otherwise, solve for x in the range $0 \leq x \leq 2\pi$ the equation

$$\cos^3 2x + 8\sin^6 x = 1.$$

[5 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2018 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	4th September 2018
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x - \frac{1}{x}\right) y = x^2,$$

given that $y = 2$ when $x = 1$.

[8 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9y = 2x^2 - 5,$$

given that $y = \frac{32}{81}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

[7 marks]

2. (a) Show that the equation $e^x = \ln(x^2 + 1) + 5$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 2 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[8 marks]

(b) Evaluate $\int_0^{\pi/2} \frac{1}{1 + \sin^2 x}\, \mathrm{d}x$ by Simpson's Rule with an interval width of $h = \frac{\pi}{8}$. Give your answer to **four** decimal places.
[Note that angles should be taken in radians.]

[7 marks]

3. (a) (i) Let $I_n = \int_{1/2}^{1} x^n e^{2x}\, \mathrm{d}x$. Show that

$$I_n = \frac{1}{2}\left(e^2 - \frac{e}{2^n} - nI_{n-1}\right).$$

(ii) The region bounded by the curve $y = x^{3/2}e^x$ and the x-axis between $x = \frac{1}{2}$ and $x = 1$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[3, 5 marks]

(b) (i) Let $y = \frac{1}{2}(e^x + e^{-x})$. Show that $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2$.

(ii) The part of the curve given by $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 1$ and $x = 2$ is rotated by 2π radians about the x-axis. Show that the area of the surface of revolution so formed is given by

$$\frac{\pi}{4}\left(4 + e^4 - e^2 - e^{-4} + e^{-2}\right).$$

[2, 5 marks]

4. (a) Given that x and y are real numbers, solve the equation

$$\frac{x}{1+i} + \frac{y}{1-2i} = i.$$

[3 marks]

(b) (i) Show that when $z \neq \pm 1$,

$$1 + z^2 + z^4 = \frac{z^6 - 1}{z^2 - 1}$$

and hence, by letting $z = e^{i\theta}$, prove that when θ is not an integral multiple of π

$$e^{i\theta} + e^{3i\theta} + e^{5i\theta} = \frac{\sin 6\theta}{2\sin\theta} + i\frac{\sin^2 3\theta}{\sin\theta}.$$

(ii) Deduce that

$$\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta.$$

(iii) Hence, evaluate

$$\int_0^{\pi/9} \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}\, d\theta.$$

[8, 1, 3 marks]

5. The function f is given by $f(x)=\dfrac{(x+2)^2}{x^2-9}$.
   (a) Find where the curve $y=f(x)$ cuts the coordinate axes.

[2 marks]

   (b) Determine the equations of the asymptotes of the curve $y=f(x)$.

[2 marks]

   (c) Find the coordinates of the stationary points of the curve.

[3 marks]

   (d) Sketch the curve of $y=f(x)$.

[4 marks]

   (e) On the same set of axes, sketch the graph of the curve $y=\dfrac{1}{f(x)}$.

[4 marks]

6. The curve $\mathscr{C}_1$ has polar equation $r=3-2\cos\theta$, where $0\le\theta\le 2\pi$.
   (a) Sketch the curve $\mathscr{C}_1$.

[5 marks]

The circle $\mathscr{C}_2$ has radius 2 and centre at the pole O.
   (b) Sketch $\mathscr{C}_2$ on the same diagram used in (a).
   (c) Find the polar coordinates of the points of intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$.

[2, 2 marks]

From the two points found in (c), let P be the point with polar coordinates $(2,\alpha)$ such that $0<\alpha<\dfrac{\pi}{2}$, and let Q be the other point. A straight line passing through P and the pole O intersects $\mathscr{C}_1$ again at the point R.
   (d) Find the polar coordinates of the point R.
   (e) Show that QR is a tangent to the circle $\mathscr{C}_2$.

[1, 5 marks]

7. (a) The function f is defined by $f(x)=\dfrac{9+4x^2}{(1-2x)^2(2+x)}$.
      (i) Express $f(x)$ in partial fractions.
      (ii) Find the expansion of $f(x)$ as a series of ascending powers of x up to and including the term in x^3.
      (iii) State the values of x for which the expansion is valid.

[3, 4, 1 marks]

   (b) (i) Explain why $n(n+1)$ is divisible by 2 for every integer $n\ge 1$.
      (ii) Prove by the principle of mathematical induction that $n(n+1)(n+2)$ is divisible by 6 for every integer $n\ge 1$.
      [Hint: Use (b)(i) in the inductive step.]

[1, 6 marks]

8. The points A, B and C have position vectors $\mathbf{i}-3\mathbf{j}+2\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ and $4\mathbf{i}-\mathbf{j}-3\mathbf{k}$, respectively.
   (a) Find the equation of the plane Π_1 that passes through A, B and C.
   [5 marks]
   (b) Find the equation of the plane Π_2 that is perpendicular to the vector $\overrightarrow{AB}$ and passes through the point D with position vector $4\mathbf{i}-5\mathbf{j}+2\mathbf{k}$. Show that the point C lies on this plane, and find the vector equation of the line of intersection of the two planes.
   [5 marks]
   (c) Find the distance of D from the plane Π_1, and the volume of the tetrahedron $ABCD$.
   [5 marks]

9. (a) **A** and **X** are the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ respectively, where $b \neq 0$.
   (i) Prove that if **A** commutes with **X** (i.e. **AX** = **XA**), then $u = \dfrac{cy}{b}$ and $v = x + \dfrac{(d-a)y}{b}$.
   (ii) Let **R** be the 2×2 matrix corresponding to an anticlockwise rotation about the origin through an angle θ, where $0 < \theta < \pi$. Find **R**.
   Show that if the matrix $\mathbf{M} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ commutes with **R** then
   $$\mathbf{M} = (p + q\cot\theta)\mathbf{I} - q\operatorname{cosec}\theta\,\mathbf{R},$$
   where **I** is the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
   [4, 3 marks]
   (b) Let Π_1, Π_2 and Π_3 be the planes with Cartesian equations $x + 2y + 3z = 5$, $kx - y + 2z = 7$ and $3x - 3y + 9z = 10$ respectively, where k is a constant.
   (i) Find k given that Π_1, Π_2 and Π_3 do not have a unique point of intersection.
   (ii) Let $k = 2$. Find the coordinates of the point of intersection of Π_1, Π_2 and Π_3.
   [3, 5 marks]

10. (a) A beverages company conducted a survey with the students following a course at a university to determine the probability that they would try a new softdrink and a new energydrink that the company was about to introduce in the market. From the respondents, 75% said that they would try the softdrink, and of these, 25% said that they were also ready to try the energydrink. From the rest of the respondents, only 10% said that they were ready to try the energydrink. What percentage of the students surveyed were ready to try the energydrink?
   [5 marks]
   (b) How many 3–digit numbers are there between 000 and 999, both inclusive, that have at least two neighbouring digits which are the same? (Note that neighbouring digits are digits that are next to each other in the number.) How many of these are even numbers?
   [5, 5 marks]