MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2020 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	10th September 2020
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Use partial fractions to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y(y+1)}{x(x-1)} = 0,$$

giving your answer in the form $y = f(x)$.

[10 marks]

2. (a) Let $y = (x^2+1)e^{2x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2e^{2x}.$$

[5 marks]

(b) A curve has equation $y^2 - 3xy + x^2 + 11 = 0$. Find the equation of the tangent line to the curve at the point $(3, 4)$.

[5 marks]

3. The lines ℓ_1 and ℓ_2 have vector equations

$$\mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + \alpha\mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j})$$
$$\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

respectively.

(a) Find α given that the lines intersect, and find the position vector of the point of intersection P.

[4 marks]

(b) Let A be the point on ℓ_1 that corresponds to $\lambda = 1$ and let B be the point on ℓ_2 that corresponds to $\mu = -1$. Find the angle $\angle APB$ and the distance between A and B.

[6 marks]

4. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = e^{2\sin x} + 2$.

(a) Find the range of f.

[2 marks]

(b) (i) Explain why f is not an injection (i.e. one-one).

(ii) State whether or not f is a surjection (i.e. onto). Give reasons for your answer.

[1, 1 marks]

(c) A new function g is now defined as follows: $g : \left[\frac{\pi}{2}, k\right] \to A$, where $g(x) = e^{2\sin x} + 2$ and $k \geq \frac{\pi}{2}$.

(i) Find the maximum value of k for which g is an injection. For this value of k, what values can A take to make $g(x)$ a bijection (i.e. one-one and onto)?

(ii) For the values of k and A found in part (i), find an expression for the inverse function $g^{-1}(x)$ and state its domain.

[3, 3 marks]

5. (a) The first term in a progression is 36 and the second term is 32.

(i) Given that the progression is geometric, find the sum to infinity.

(ii) Given instead that the progression is arithmetic, find the number of terms in the progression if the sum of all the terms is 0.

[2, 3 marks]

(b) Let $i = \sqrt{-1}$. Given that

$$z = \frac{\cos 8\theta + i \sin 8\theta - 1}{\sin 4\theta + 2i - 2i \sin^2 2\theta}, \quad \left(\theta \neq k\frac{\pi}{4} \text{ for } k \in \mathbb{Z}\right)$$

show that $|z| = 2|\sin 2\theta|$ and $\arg z - 6\theta = 2n\pi$ for some $n \in \mathbb{Z}$.

[5 marks]

6. Let $f(x) = 5x^3 + 8x^2 + 2x + 12$, where $x \in \mathbb{R}$.
   (a) Show that $f(x)$ can be written in the form $(x+2)g(x)$ where $g(x)$ is a quadratic expression satisfying $g(x) > 0$ for every $x \in \mathbb{R}$.

[5 marks]

   (b) Solve the inequality $f(x) > 0$.

[1 marks]

   (c) Hence, or otherwise, find all values of θ in $[0, \pi] \setminus \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ for which

$$5\tan^3 2\theta + 8\tan^2 2\theta + 2\tan 2\theta + 12 > 0\,.$$

[4 marks]

7. (a) Use an appropriate trigonometric identity to evaluate the integral

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\; \mathrm{d}x\,.$$

[5 marks]

   (b) Use integration by parts to find the integral

$$\int (x^2 - 2x + 1)e^{2x}\; \mathrm{d}x\,.$$

[5 marks]

8. (a) The coordinates (x, y) of a point P are given in terms of the parameter t by the equations $\sin t = \dfrac{y-1}{2}$ and $\tan t = \dfrac{y-1}{x-1}$. Find the Cartesian equation of the locus C_1 of P, and hence deduce that it is a circle with centre $(1, 1)$ and radius 2.

[4 marks]

   (b) A circle C_2 with centre $(a, 1)$ and radius r lies inside C_1 and touches C_1 internally at one point. Determine the possible equations of C_2, distinguishing between the ranges of values of a for which each equation is valid.

[6 marks]

9. Let the matrix $\mathbf{A}$ be given by $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d can randomly take any value from $\{-\sqrt{2}, 0, \sqrt{2}\}$.
   (a) How many different matrices $\mathbf{A}$ can be formed?

   [2 marks]

   (b) What is the probability that the matrix formed is invertible?

   [5 marks]

   (c) What is the probability that the matrix formed is invertible and has at least one entry equal to zero?

   [3 marks]

10. (a) Let $f(x) = (\sqrt{3} - 2\sin\theta)\,x^2 + x\sin\theta + (\sqrt{3} + 2\sin\theta)$. Giving your answers in degrees and correct to 2 decimal places find:
    (i) all real values of θ such that the graph $y = f(x)$ is a straight line;
    (ii) all real values of θ for which the equation $f(x) = 0$ has exactly one real solution.

    [2, 3 marks]

    (b) Show that for every real value u, the quadratic equation

$$(2e^{u} - 1)x^2 + 3e^{u}x + (e^{u} - 1) = 0$$

    has real solutions.

    [5 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2020 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	12th September 2020
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Solve the first order linear differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin x\,, \qquad \text{where } 0 \le x < \frac{\pi}{2}\,,$$
given that $y = 2$ when $x = 0$. Give your answer in the form $y = f(x)$.

   (b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} = 2\sin x\,,$$
given that $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

[7, 8 marks]

2. (Note that angles should be taken in radians throughout this question.)

   (a) Show that the equation $3\ln(x^2+1) = \cos 2x + 1$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[7 marks]

   (b) (i) Obtain an approximate value of $\ln 1.2$ by finding the integral of the function $f(x) = \frac{1}{x}$ from 1 to 1.2 using the Trapezoidal Rule with an interval width of $h = 0.05$. Give your answer to **four** decimal places.

   (ii) Write down the series expansion of $\ln(1+x)$ up to and including the term in x^4. Use this series expansion to obtain an approximate value of $\ln 1.2$. Give your answer to **four** decimal places.

[4, 4 marks]

3. (a) (i) Let $I_n = \int x^n \sqrt{1+x}\,\mathrm{d}x$. Show that

$$I_n = \frac{2}{2n+3} x^n (1+x)^{\frac{3}{2}} - \frac{2n}{2n+3} I_{n-1}\,.$$

(ii) The region bounded by the curve $y = 3x(1+x)^{\frac{1}{4}}$ and the x-axis between $x = 1$ and $x = 3$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[5, 4 marks]

(b) (i) Show that

$$\left(1 + \frac{1}{t^{\frac{2}{3}}}\right)^{\frac{1}{2}} = t^{-\frac{1}{3}} \left(1 + t^{\frac{2}{3}}\right)^{\frac{1}{2}}\,.$$

(ii) A curve is given parametrically by $x = 2t$ and $y = 3t^{\frac{2}{3}}$. Find the length of the arc of the curve from the point where $t = 1$ to the point where $t = 8$.

[1, 5 marks]

4. (a) (i) Express $\dfrac{5r+13}{(r+2)(r+3)(r+5)}$ into partial fractions.

(ii) Hence, evaluate

$$S_n = \sum_{r=1}^{n} \frac{5r+13}{(r+2)(r+3)(r+5)}\,.$$

(iii) Deduce S_∞.

[2, 5, 1 marks]

(b) In a fashion retail outlet, 90% of the items are equipped with an activated security tag that is automatically scanned by an alarm system installed at the door, while the others have a fake security tag. There is 98% chance that an item which has not been paid for triggers the alarm, whereas there is a 7% chance that the alarm is triggered by a duly paid item. From previous surveys, it is known that 3% of the people entering the outlet try to exit without paying for an item (it can be assumed that this percentage is independent of the number of items carried by a person). What is the probability that:

(i) A person carrying one item does not trigger the alarm while exiting the outlet?

(ii) A person carrying three items triggers the alarm while exiting?

[3, 4 marks]

5. The curve $\mathscr{C}$ has polar equation $r = f(\theta)$ for $0 \le \theta \le 2\pi$, where $f(\theta) = 2 + \dfrac{7}{2}\cos 4\theta$.
   (a) Show that
      (i) $f(\theta) = f(-\theta)$, and
      (ii) $f(\frac{\pi}{2} - \theta) = f(\frac{\pi}{2} + \theta)$.
   Hence, deduce that $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are lines of symmetry of $r = f(\theta)$.
   [3 marks]
   (b) Find the equations of the tangents to the curve at the pole for $0 \le \theta \le \frac{\pi}{2}$.
   [3 marks]
   (c) By using the symmetry properties of $r = f(\theta)$ and taking values of θ at intervals of $\frac{\pi}{12}$ for $0 \le \theta \le \frac{\pi}{2}$, or otherwise, sketch the curve $\mathscr{C}$.
   [4 marks]
   (d) Find the area enclosed within all the loops of the curve $\mathscr{C}$.
   [5 marks]

6. The planes Π_1 and Π_2 have equations $3x + y - 4z = 10$ and $x + y - z = 5$, respectively:
   (a) Find the point A that lies on both planes and has x-coordinate equal to 4. Find the equation of the line ℓ_1 where the two planes intersect.
   [6 marks]
   (b) Find the equation of the plane Π_3 that contains the line ℓ_1 and passes through the origin.
   [4 marks]
   (c) The points B and C have coordinates $(2, -3, 2)$ and $(1, 4, -1)$ respectively. Find the area of the triangle ABC.
   [5 marks]

7. In this question $i = \sqrt{-1}$ and $x, y \in \mathbb{R}$. Let $P(x, y)$ be the point on an Argand diagram representing the complex number $z = x + iy$ and satisfying the equation

$$|z + a| = k|z + ib|,$$

   where a, b and k are constant real numbers, and $k \ge 0$.
   (a) Describe the locus of $P(x, y)$ in the case when $k = 0$ and in the case when $k = 1$.
   [3 marks]
   (b) Show that if $k \ne 0$ and $k \ne 1$, then the locus of $P(x, y)$ is a circle. Find the coordinates of the centre and the radius of the circle in terms of a, b and k.
   [8 marks]
   Hence, or otherwise, show that $\dfrac{5}{24}(\sqrt{101} + \sqrt{29})$ is the greatest value of $|z|$ when

$$\left|\frac{z + 5i}{z - 2}\right| = 5.$$

   [4 marks]

8. (a) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 .$$

[7 marks]

(b) The function f is defined by $f(x) = xe^{2x}$. Prove by the principle of mathematical induction that for every integer $n \geq 1$,

$$f^{(n)}(x) \equiv \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(xe^{2x}\right) = 2^{n-1}(2x+n)e^{2x} ,$$

where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

[8 marks]

9. (a) Let a, b and c be real numbers and consider the following system of linear equations in unknowns x, y and z:

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c . \end{aligned} \qquad (\star)$$

(i) Show that the system of equations $(\star)$ does not have a unique solution, irrespective of the values of a, b and c. What condition must be placed on a, b and c so that $(\star)$ has a solution?

(ii) Solve the equations when $a = c = 1$ and $b = 2$.

[5, 5 marks]

(b) Find all matrices $\mathbf{B} = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$ that commute with $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$; that is $\mathbf{AB} = \mathbf{BA}$.

[5 marks]

10. The function f is given by $f(x) = \dfrac{6x^2 - 7x + 2}{2x^2 + 7x + 3}$.

(a) Determine the equations of the asymptotes of the curve $y = f(x)$.

[3 marks]

(b) Find where the curve $y = f(x)$ cuts the coordinate axes.

[2 marks]

(c) Find the stationary points of the curve $y = f(x)$ and determine their nature.

[4 marks]

(d) Sketch the curve of $y = f(x)$.

[4 marks]

(e) Hence, show that the equation $f(x) = x + 3$ has only one real root by marking this root on your graph.

[2 marks]