MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2020 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	14th December 2020
TIME:	16:00 to 19:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Use an appropriate substitution to find the integral

$$\int \frac{1+\ln x}{x}\,\mathrm{d}x\,.$$

[3 marks]

(b) Hence, find the general solution of the differential equation

$$x(y+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 1+\ln x\,.$$

Find also the particular solution for which $y=-1$ when $x=1$.

[5, 2 marks]

2. (a) Let $y = x^2 \ln x$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2\,.$$

[5 marks]

(b) A curve has equation $2y^2 - 3x^2y + x^3 - 2 = 0$. Find the equation of the tangent line to the curve at the point $(3, 1)$.

[5 marks]

3. The point A has position vector $5\mathbf{i}-\mathbf{j}-6\mathbf{k}$, and the line ℓ_1 has vector equation

$$\mathbf{r}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

   (a) Find the point B on ℓ_1 such that $\overrightarrow{AB}$ is perpendicular to ℓ_1. Hence, write down the equation of the line ℓ_2 that passes through A, intersects ℓ_1, and is perpendicular to ℓ_1.

[5 marks]

   (b) The vector $\alpha\mathbf{i}+\beta\mathbf{j}+\mathbf{k}$ is perpendicular to both ℓ_1 and ℓ_2. Find α and β.

[3 marks]

   (c) Find the distance AB.

[2 marks]

4. Let $f(x)=(1+x^6)\left(4x^2-\dfrac{1}{2x}\right)^9$, where $x\in\mathbb{R}\setminus\{0\}$.

   (a) Find the coefficient of x^3 in the expansion of $f(x)$.

[5 marks]

   (b) Solve the inequality $f(x)\geq 0$.

[5 marks]

5. (a) Find the locus of the point P which is equidistant from the point A with coordinates $(4,-1)$ and the line with equation $y=2x$.

[3 marks]

   (b) Determine the coordinates of the two points B and C where the locus in part (a) intersects the line with equation $3x+y-4=0$.

[3 marks]

   (c) Determine the image of the line BC under the linear transformation represented by the matrix $\mathbf{M}=\begin{pmatrix}-\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}$.

[4 marks]

6. (a) Use partial fractions to find

$$\int\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,\mathrm{d}x\,.$$

[5 marks]

   (b) Use the substitution $x=3\sin\theta$ to find

$$\int\frac{x}{\sqrt{9-x^2}}\,\mathrm{d}x\,.$$

[5 marks]

7. (a) Resolve into partial fractions the function

$$f(x) = \frac{x^2 - 1}{2x^2 - 13x + 6}.$$

[4 marks]

(b) Show that

$$\frac{1}{\cos x} - \frac{\cos x}{1 + \sin x} \equiv \tan x$$

and hence, solve the equation

$$\frac{1}{\cos x} - \frac{\cos x}{1 + \sin x} + 2 = 0$$

for $0^\circ \leq x \leq 360^\circ$.

[3 marks]

(c) Solve the equation

$$2\log_9 x + 3 = 2\log_x 9.$$

[3 marks]

8. Robert rolls a fair 12-sided die three times. Each face of the die has one of the numbers from 1 to 12 such that no face has the same number. The number on the top of the die after each roll is recorded.

(a) If the numbers are written down in the exact order in which they appear, how many different ordered triples are possible?

[2 marks]

(b) How many of the ordered triples would be made up of:

(i) the numbers 1, 1 and 2;

(ii) the numbers 1, 2 and 3?

[2 marks]

(c) Hence, or otherwise, if the order in which the numbers are drawn is not important, how many different unordered triples are possible?

[4 marks]

(d) If the values of each triple are added and the sum obtained is recorded, how many different sums are possible?

[2 marks]

9. (a) Let $i = \sqrt{-1}$. Given that $z = 1 + \sqrt{3}i$, express $\dfrac{1}{\sqrt{z}}$ in the form $p + qi$, where p and q are real numbers given in surd form.

[5 marks]

(b) How many integers greater than or equal to zero and less than 1000 are not divisible by 2 or 5? What is the average value of these integers?

[5 marks]

10. (a) The function $f(x) = \dfrac{x}{x+2}$ is defined for all real $x \neq -2$. If $g(x) = f\left(\dfrac{x-2}{3}\right)$, derive an expression for $g(x)$ and find its domain.

[4 marks]

(b) The functions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are defined by $p(x) = \sqrt[3]{2x-1}$ and $q(x) = \dfrac{1}{2}\left(x^3+1\right)$. Find an expression for the composite functions $(p \circ q)(x)$ and $(q \circ p)(x)$, stating their domain and range (as composite functions). Determine whether p and q are mutual inverses.

[6 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2020 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	15th December 2020
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the first order linear differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \frac{x^3}{x^2+1},$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$.

[8 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2e^x,$$

given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.
[When finding the particular integral, use Ax^2e^x as a trial solution.]

[7 marks]

2. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -1 \\ 5 & 12 & -5 \end{pmatrix}$. Find all vectors $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\mathbf{Av} = \mathbf{0}$.

[7 marks]

(b) Let $\mathbf{P} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ satisfy $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}\mathbf{u} = \mathbf{u}$, where $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find a, b and c, given that $c \neq 0$.

[8 marks]

3. (Note that angles should be taken in radians throughout this question.)
   (a) Show that the equation $e^{\cos x} = x^3$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

   [8 marks]

   (b) Evaluate $\displaystyle\int_0^{\pi} e^{\cos x}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = \pi/4$. Give your answer to **four** decimal places.

   [7 marks]

4. (a) (i) Show that $\displaystyle\int \ln x\,\mathrm{d}x = x\ln x - x + C$, where C is a constant.
   (ii) Let $I_n = \displaystyle\int (\ln x)^n\,\mathrm{d}x$. Show that $I_n = x(\ln x)^n - nI_{n-1}$.
   (iii) The region bounded by the curve $y = (\ln x)^2$ and the x-axis between $x = 1$ and $x = e$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

   [2, 3, 5 marks]

   (b) A cylindrical shaped metal bar is expanding because it is being heated. The length of the bar at any point in time is $5r$, where r is the radius of the circular cross-section. The area of this cross-section is increasing at a rate of 0.06 cm^2/sec when the radius is 4 cm. Find the rate of change of the volume of the bar at this instant.

   [5 marks]

5. Let $i = \sqrt{-1}$.
   (a) (i) Using Euler's identity $e^{i\theta} \equiv \cos\theta + i\sin\theta$, prove the compound angle identities:

$$\begin{aligned}\cos(x+y) &\equiv \cos x\cos y - \sin x\sin y,\\ \sin(x+y) &\equiv \sin x\cos y + \cos x\sin y.\end{aligned}$$

   (ii) Show that if $z = \cos\theta + i\sin\theta$ and $p \in \mathbb{R}$, then

$$2\cos p\theta = z^p + z^{-p} \quad\text{and}\quad 2i\sin p\theta = z^p - z^{-p}.$$

   (iii) Prove that

$$16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta,$$

   for every $\theta \in \mathbb{R}$. Hence, evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5\theta\,\mathrm{d}\theta.$$

   [2, 2, 7 marks]

   (b) Find each of the fourth roots of $8\sqrt{3} + 8i$.

   [4 marks]

6. The function f is given by $f(x) = \dfrac{2x-5}{2x^2-7x+3}$.
   (a) Determine the equations of the asymptotes of the curve $y = f(x)$.
   [3 marks]
   (b) Find where the curve $y = f(x)$ cuts the coordinate axes.
   [2 marks]
   (c) Sketch the curve of $y = f(x)$.
   [4 marks]

   The function g is given by $g(x) = \dfrac{2x^2-7x+3}{2x-5}$.
   (d) Find the equation of the oblique asymptote of $y = g(x)$.
   [3 marks]
   (e) Hence, sketch the curve of $y = g(x)$ on the same diagram used for $y = f(x)$.
   [3 marks]

7. (a) (i) Obtain the Maclaurin's series of the function $(2-\sin x)^2$, up to and including the term in x^5.
   (ii) Give the coefficient of x^{2k+1} in this expansion, where k is any positive integer.
   [4, 2 marks]
   (b) (i) Show that for any positive integer n,
   $$1+2+3+4+\ldots+(2^{n-1}-1) = 2^{n-2}\left(2^{n-1}-1\right).$$
   (ii) Find the sum of the even numbers between, but **not** including, any two consecutive powers of 2, (that is, between 2^n and 2^{n+1}, **not** including both; for example, between 2^1 and 2^2, or between 2^2 and 2^3).
   (iii) By first finding the sum of all the integers between, but **not** including, any two consecutive powers of 2, deduce the sum of **all** the odd numbers between two consecutive powers of 2.
   [2, 4, 3 marks]

8. The points A, B and C have position vectors $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $5\mathbf{i}+2\mathbf{k}$ respectively.
   (a) Find the volume of the tetrahedron whose vertices are the origin O and the points A, B and C.
   [3 marks]
   (b) The points A, B and C lie on the plane Π_1. Find the equation of Π_1.
   [5 marks]
   (c) Find the equation of the line ℓ_1 that passes through the midpoints of the line segments AB and AC. Does ℓ_1 intersect the line that passes through the points B and C? Explain.
   [5 marks]
   (d) Find the angle between the line segments OA and OB.
   [2 marks]

9. Prove by the principle of mathematical induction that for every integer $n \geq 1$:
   (a) $13^n - 4^n$ is divisible by 9.

[7 marks]

   (b)

$$\frac{1}{4 \cdot 1^2 - 1} + \frac{1}{4 \cdot 2^2 - 1} + \frac{1}{4 \cdot 3^2 - 1} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}.$$

[8 marks]

10. A phone-store carried out a survey with 250 people to investigate their preference for three different models of mobile phones: Model A, Model B and Model C.
    - 9 people liked all three models.
    - 100 people liked Model A.
    - There were three times as many people who liked Models A and C but not B than there were who liked Models A and B but not C.
    - Those who like Models A and C but not B were five times as many as those who liked all three models.
    - 150 people liked Model C.
    - 25 people did not like any model.
    - The group of people who liked Models A and C but not B was exactly 50 less than the group who liked both Models B and C.

    (a) Draw a Venn diagram to represent the result of the survey.

[6 marks]

    (b) A person is chosen at random from the respondents. What is the probability that this person:
      (i) liked Model B;
      (ii) liked Model B given that he/she liked Model A?

[2, 2 marks]

    (c) Three different people are to be chosen at random from the respondents to be given a gift as an appreciation for their participation. What is the probability that:
      (i) all three would have liked all the three models;
      (ii) exactly one would have liked none of the models?

[2, 3 marks]