MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2021 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	12th June 2021
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Solve the differential equation

$$(3+x)(3x+1)\cos y\,\mathrm{d}y - 2x(1+\sin y)\mathrm{d}x = 0,$$

giving your answer in the form $|1+\sin y|^p = \dfrac{c|3+x|^q}{|1+3x|}$, where c is a constant. Give the values of p and q.

[10 marks]

2. (a) Let $y = e^{-\left(x^{\frac{3}{2}}\right)}$. Show that

$$4x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 9x^2 y = 0.$$

[5 marks]

(b) A curve has equation $2y^2 + 4xy - 3x^2 = 3$. Find the equation of the normal to the curve at the point $(-1, 3)$.

[5 marks]

3. The point A has position vector $\mathbf{i}+5\mathbf{k}$ and the line ℓ_1 has vector equation

$$\mathbf{r}=7\mathbf{i}+4\mathbf{j}-2\mathbf{k}+\lambda(5\mathbf{i}+3\mathbf{j}+\mathbf{k}).$$

(a) Find the equation of the line ℓ_2 that passes through A and is perpendicular to ℓ_1.

[7 marks]

(b) Find the distance between A and the point B where the lines ℓ_1 and ℓ_2 intersect.

[3 marks]

4. (a) The real functions f and g are defined by

$$f(x)=\frac{x}{x+1} \quad \text{and} \quad g(x)=\frac{1}{x-2}.$$

Determine the composite function $f\circ g$ and state its domain.

[5 marks]

(b) A function f is defined by $f(x)=\dfrac{5}{2x}+7$ for real values of $x\neq 0$. Define the inverse function f^{-1} stating its domain and range. Hence, find also the range of f.

[5 marks]

5. (a) Find a and b given that

$$\frac{3+7i}{10+4i}=a+ib.$$

Given that $1/(a+ib)$ is a root of the equation $z^2+\alpha z+\beta=0$, find the real numbers α and β.

[5 marks]

(b) Express $\sqrt{3}\sin\theta-4\cos\theta$ in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0^\circ\leq\alpha\leq 90^\circ$, giving the exact value of R and the value of α correct to 2 decimal places. Find the smallest positive angle θ at which $\sqrt{3}\sin\theta-4\cos\theta$ takes its maximum value.

[5 marks]

6. (a) If $x+4$ is a factor of $f(x)=6x^3+kx^2-19x-60$, find the value of k and express $f(x)$ as a product of linear factors.

[5 marks]

(b) Show that $x\log_4 32+y\log_8 4=\dfrac{5x}{2}+\dfrac{2y}{3}$. Hence, solve the following two simultaneous equations:

$$x\log_4 32+y\log_8 4=\frac{17}{6}$$
$$\log_2 x^3+\log_2 y=2\log_2 x.$$

[5 marks]

7. (a) Use an appropriate substitution to find the integral $\displaystyle\int \frac{x^2}{x^3+1}\,\mathrm{d}x$.

[4 marks]

(b) Use integration by parts to find the integral $\displaystyle\int e^{3x}\cos x\,\mathrm{d}x$.

[6 marks]

8. (a) Expand $\left(1-\frac{x}{2}\right)^{\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^3. Use this expansion to find $\sqrt[3]{999}$ correct up to 4 decimal places.

[6 marks]

(b) What is the least number of terms that must be taken from of the geometric series $2+10+50+250+1250+\ldots$ so that the sum exceeds one million?

[4 marks]

9. (a) Find the image of the points with coordinates $(1,0)$ and $(0,1)$ under a clockwise rotation by a non-reflex angle α about the origin. Hence, deduce the matrix **P** representing the given linear transformation in the plane.
(b) Deduce the matrix **Q** representing an anticlockwise rotation by a non-reflex angle β about the origin.
(c) When are the two matrices **P** and **Q** the inverses of each other?
(d) Given that $\alpha > \beta$, find the image of the point having coordinates $(1,1)$ after undergoing the transformation **P** followed by the transformation **Q** (giving your answer in its simplest form).
(e) Would the image of the point $(1,1)$ be the same if the order of the transformations described above is reversed? Explain briefly why.

[3, 1, 1, 3, 2 marks]

10. (a) The graph of $y = f(x)$ cuts the y-axis at the point $(0,b)$, where b is a constant. For any constant a, where would the graphs of the following three functions cut the y-axis?
(i) $y = f(x)+a$;
(ii) $y = af(x)$;
(iii) $y = a(f(x)+a)$.

[1, 1, 1 marks]

(b) (i) How many different 6-digit binary numbers (that is, using only the digits 0 and 1) can be formed?
(ii) A binary number is called a palindrome if it reads the same backward and forward. How many 6-digit binary numbers are palindrome?
(iii) What is the probability that a 6-digit binary number chosen at random is a palindrome starting with the digit 1.

[2, 2, 3 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	14th June 2021
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the first order linear differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+3y=4x^2-3x\,, \qquad \text{where } x>0\,,$$

given that $y=1$ when $x=1$. Give your answer in the form $y=f(x)$.

[7 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+5y=10x^2-3x-3\,,$$

given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=4$ when $x=0$.

[8 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $e^{-x^2}=x$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[8 marks]

(b) Evaluate the integral $\displaystyle\int_0^1 \ln(1+\cos x)\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=0.25$. Give your answer to **four** decimal places.

[7 marks]

3. (a) (i) Show that $\cot^2 x + 1 = \operatorname{cosec}^2 x$.

(ii) Let $I_n = \int \cot^n x \, \mathrm{d}x$. Show that

$$(n-1)I_n = -\cot^{n-1} x - (n-1)I_{n-2}.$$

[Hint: write $\cot^n x$ as $\cot^{n-2} x \cot^2 x$.]

(iii) The region bounded by the curve $y = \cot^{5/2} x$ and the x-axis between $x = \pi/6$ and $x = \pi/4$ is rotated through 2π radians about the x-axis. Show that the volume of the solid that is generated by this rotation is $(1 + \frac{\ln 2}{2})\pi$.

[2, 5, 4 marks]

(b) A curve is given parametrically by $x = t - 2t^2$ and $y = \frac{8}{3}t^{3/2}$. Find the length of the arc of the curve from the point where $t = 2$ to the point where $t = 5$.

[4 marks]

4. (a) (i) Express $\dfrac{3x+4}{x^3+3x^2+2x}$ into partial fractions.

(ii) Determine $S_n = \sum_{r=1}^{n} \dfrac{3r+4}{r^3+3r^2+2r}$ and S_∞.

(iii) Evaluate

$$\frac{10}{2\cdot3\cdot4} + \frac{13}{3\cdot4\cdot5} + \frac{16}{4\cdot5\cdot6} + \cdots + \frac{34}{10\cdot11\cdot12}.$$

[2, 6, 2 marks]

(b) Given that $f(x) = \dfrac{1}{(x+1)(x+2)}$ and its n^{th} derivative is given by

$$f^{(n)}(x) = n!\left(\frac{1}{(-x-2)^{n+1}} - \frac{1}{(-x-1)^{n+1}}\right),$$

(i) find the coefficient of x^n in the Maclaurin's series of $f(x)$; and

(ii) write down the first **four** terms of this series.

[2, 3 marks]

5. (a) Obtain an expression for $\sum_{r=1}^{n}(3r-2)^2$ in terms of n. Hence, find

$$4^2+7^2+10^2+13^2+\ldots+28^2.$$

[3, 3 marks]

(b) The circle $\mathscr{C}$ has Cartesian equation given by $(x-1)^2+(y-4)^2=17$.

(i) Write the equation of $\mathscr{C}$ in polar form.

(ii) Find the area of the region bounded by the circle $\mathscr{C}$, the initial line and the half-line $\theta=\dfrac{\pi}{2}$.

[3, 6 marks]

6. The points A, B and C have position vectors $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}-3\mathbf{k}$, respectively.

(a) Find the volume of the parallelepiped with sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

[3 marks]

(b) Find the equation of the plane Π containing A, B and C.

[6 marks]

(c) The line ℓ passes through the origin and is perpendicular to Π. Find the point of intersection D of ℓ with Π. Find also the distance of Π from the origin.

[4, 2 marks]

7. (a) Consider the following two quadratic functions of $z\in\mathbb{C}$,

$$f(z)=z^2+az+13$$
$$g(z)=z^2+bz+2,$$

where a and b are positive integers.

(i) Find **all** the possible values that a and b can take given that $f(z)-g(z)$ is constant and the equation $f(z)\cdot g(z)=0$ has no real solution.

(ii) By considering the expression for $f(z)\cdot g(z)$, solve the equation

$$z^4+4z^3+19z^2+30z+26=0,$$

given that it has no real roots. Find the modulus and argument of each root.

[4, 4 marks]

(b) Express $\sin x$ and $\cos x$ in terms of e^{ix} and e^{-ix}. Hence, find the value of k for which

$$\sin^5 x\cos^5 x=\frac{1}{k}(\sin 10x-5\sin 6x+10\sin 2x).$$

[7 marks]

8. (a) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{k=1}^{n}(-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}.$$

[5 marks]

(b) Show that for any real $x \geq 1$

$$\frac{1}{\sqrt{3(x-1)+1}} \cdot \frac{2x-1}{2x} \leq \frac{1}{\sqrt{3x+1}}.$$

[5 marks]

(c) Use (b) and the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

[5 marks]

9. Consider the following system of equations:

$$\begin{aligned} 3x + 2y + z &= \lambda_1 \\ x + ay + bz &= \lambda_2 \\ -x - 2y - 3z &= \lambda_3. \end{aligned}$$

(a) Show that if $a = b = 1$, then the equations are consistent only when $\lambda_1 - 4\lambda_2 - \lambda_3 = 0$.

[6 marks]

(b) Let $a = -1$ and $b = -2$.

(i) Show that, in this case the three equations have a unique solution and find x, y, z in terms of λ_1, λ_2 and λ_3.

(ii) Determine x, y, z given that $\lambda_1 - 2\lambda_2 + \lambda_3 = 0$, $\lambda_1 = 5$ and $\lambda_3 = -3\lambda_2$.

[6, 3 marks]

10. The function f is given by $f(x) = \dfrac{x^2 - 9}{4x - 3}$.

(a) Determine the equations of the vertical and oblique asymptotes of the curve $y = f(x)$.

[4 marks]

(b) Find where the curve $y = f(x)$ cuts the coordinate axes.

[3 marks]

(c) On the same diagram, sketch the curve of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

[6 marks]

(d) Hence, deduce that the equation $\left(f(x)\right)^2 = 1$ has two positive real roots and two negative real roots. Mark these roots on your graph.

[2 marks]