MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2022 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	7th May 2022
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kxy - y}{x - kxy}, \quad \text{where } k = \ln 2 \text{ and } x, y > 0,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $Pxy = 2^{Qx+Ry}$, where the values of P, Q and R need to be determined.

[10 marks]

2. A curve is given by the implicit equation

$$x^2 + 2xy + 3y^2 = 18.$$

(a) Find the coordinates of the stationary points of the curve.

[5 marks]

(b) Show that, at the stationary points, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{1}{x+3y}$.

[3 marks]

(c) Hence, determine the nature of the stationary points.

[2 marks]

3. The lines ℓ_1 and ℓ_2 have vector equations

$$\mathbf{r}_1 = 4\mathbf{i}+\mathbf{j}+7\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+2\mathbf{k}), \quad \text{and}$$
$$\mathbf{r}_2 = 5\mathbf{i}-6\mathbf{j}+2\mathbf{k}+\mu(3\mathbf{i}-5\mathbf{j}-\mathbf{k}),$$

respectively.

(a) Find the point of intersection A of the two lines.

[4 marks]

(b) The point B has position vector $11\mathbf{i}+6\mathbf{j}-5\mathbf{k}$. Show that the line passing through A and B is perpendicular to both ℓ_1 and ℓ_2.

[4 marks]

(c) Find the angle made by the lines ℓ_1 and ℓ_2.

[2 marks]

4. (a) If $f(x+1) = 4x^2 + x$, find the function $f(x)$.

[4 marks]

(b) Consider the real function $f(x) = \ln(x-1)+\ln 3$ for $x > 1$.

(i) Find the inverse function $f^{-1}(x)$.

(ii) Let $g(x) = e^x$. Determine the composite function $(g \circ f)(x)$, giving your answer in the form $ax+b$, where a and b are integers. Give also the domain of $g \circ f$.

[6 marks]

5. (a) Prove the following **two** identities

$$(\sin A+\sin B)^2+(\cos A+\cos B)^2 \equiv 2\big(1+\cos(A-B)\big),$$
$$(\sin A+\sin B)(\cos A+\cos B) \equiv \sin(A+B)\big(1+\cos(A-B)\big),$$

and hence, find $\sin(A+B)$ given that $\sin A+\sin B = \frac{\sqrt{2}}{2}$ and $\cos A+\cos B = \frac{\sqrt{6}}{2}$.

[7 marks]

(b) Solve the equation $3\sin 2\theta = \tan 2\theta$ for $\theta \in (-\pi, \pi]$, giving your answer correct to **one** decimal place.

[3 marks]

6. (a) Let S_{24} denote the sum of the first 24 terms of the series

$$\ln x + \ln x^5 + \ln x^9 + \ln x^{13} + \cdots,$$

where $x > 0$. Find an expression for S_{24}, giving your answer as a single logarithm. Hence, solve the equation

$$S_{24} = 47\ln\left(\frac{12x^{12}-5}{4}\right).$$

[5 marks]

(b) Let $f(x) = kx^2+9x+k+1$. Given that the graph of $y = f(x)$ cuts the x-axis twice, find the set of possible values of k.

[5 marks]

7. (a) Show that $1+\tan\left(\frac{1}{4}\pi-\phi\right)=\frac{2}{1+\tan\phi}$.

[3 marks]

(b) Using the substitution $\theta=\frac{1}{4}\pi-\phi$, show that

$$\int_0^{\frac{1}{4}\pi}\ln(1+\tan\theta)\,\mathrm{d}\theta=\frac{1}{8}\pi\ln 2\,.$$

[Hint: One can use the fact that $\int_a^b f(x)\,\mathrm{d}x=-\int_b^a f(x)\,\mathrm{d}x$.]

[4 marks]

(c) Using the substitution $x=\tan\theta$, evaluate

$$\int_0^1 \ln(1+x)\,\mathrm{d}x\,.$$

[3 marks]

8. (a) Determine the set of values of x for which $\frac{2x-3}{x+2}>-1$.

[5 marks]

(b) Show that $2x-1$ is a factor of $2x^3+7x^2+4x-4$. Hence, express the following expression into partial fractions

$$\frac{x^2+6}{2x^3+7x^2+4x-4}\,.$$

[5 marks]

9. (a) Four red balls and six blue balls are placed in a bag. A person takes out the balls one by one at random and places them on a table in the order in which they are extracted. What is the probability that the first two red balls in the row are next to each other?

[4 marks]

(b) (i) Show that if $\mathbf{D}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$, where a and b are constants, then $\mathbf{D}^3=\begin{pmatrix}a^3 & 0\\ 0 & b^3\end{pmatrix}$.

(ii) Find the inverse of the matrix $\mathbf{P}=\begin{pmatrix}1 & -3\\ 1 & 1\end{pmatrix}$.

(iii) Let $\mathbf{D}=\begin{pmatrix}5 & 0\\ 0 & 1\end{pmatrix}$. Find the matrix $\mathbf{A}$ which satisfies the equation $\mathbf{D}^3=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

[6 marks]

10. (a) By expanding the expression $(x-2)^3$, sketch the graph of the curve $y=x^3-6x^2+12x$. Hence, or otherwise, show that the equation $x^3-6x^2+11x=0$ has only one real root.

[5 marks]

(b) Find the locus of the point P such that the sum of its distances from the points A with coordinates $(1,2)$ and B with coordinates $(2,1)$ is 2.

[5 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2022 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	9th May 2022
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the first order linear differential equation

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\sin x = 3, \qquad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$.

[7 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 24e^{4x},$$

given that $y = -1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ when $x = 0$. Give your answer in the form $y = f(x)$.
[When finding the particular integral, use Px^2e^{4x} as a trial solution.]

[8 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $2\ln(1+\sin^2 x) = \cos x$ has a solution between 0 and π. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[8 marks]

(b) Evaluate the integral $\int_0^1 \sin[\ln(x+1)]\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to **four** decimal places.

[7 marks]

3. (a) Use the substitution $x = \sin t$ to evaluate $\displaystyle\int_0^1 (1-x^2)^{\frac{3}{2}}\,\mathrm{d}x$.

[You may need to use the identity: $8\cos^4 t = 3 + 4\cos 2t + \cos 4t$.]

[3 marks]

(b) (i) Express $\dfrac{1}{1+x} + \dfrac{1}{1-x} + \dfrac{2}{1+x^2}$ as one fraction.

(ii) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{8}{1-x^4}\,\mathrm{d}x$.

[5 marks]

(c) A function y is defined by $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and show that

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(x^3 + \frac{1}{4x^3}\right)^2.$$

(ii) The part of the curve of this function y between $x = 1$ and $x = 2$ is rotated by 2π radians about the x-axis. Find the area of the surface of revolution so formed.

[7 marks]

4. (a) Find the first four derivatives of the function $f(x) = \dfrac{1}{2x+1}$.

Hence:

(i) Write down the first five terms of the Maclaurin's series of $f(x)$;

(ii) Find the decimal representation of $\frac{8}{9}$ correct to **three** decimal places.

[8 marks]

(b) (i) Show that

$$\sum_{r=3}^{n} \ln\left(\frac{r(r-1)}{(r-2)^2}\right) = \ln\left(\frac{n(n-1)^2}{2}\right).$$

(ii) Hence, find the sum of the first six terms of the series, giving your answer in terms of $2\ln a$, for some constant a to be determined.

[7 marks]

5. (a) The curve $\mathscr{C}$ has Cartesian equation given by $\left(\sqrt{x^2+y^2}\right)^3 = 4xy$.
   (i) Write the equation of $\mathscr{C}$ in polar form.
   (ii) Sketch the curve $\mathscr{C}$ for $0 \le \theta \le \pi$.
   (iii) Find the area enclosed by $\mathscr{C}$ for $0 \le \theta \le \pi$.

[9 marks]

(b) (i) Find $\sum_{r=1}^{n} (r-3)^3$.
   (ii) Hence, deduce the sum $2^3 + 3^3 + 4^3 + \cdots + 9^3$.

[6 marks]

6. The point A has position vector $3\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and the line ℓ_1 has vector equation

$$\mathbf{r} = 2\mathbf{i}-\mathbf{j}-4\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}-4\mathbf{k}).$$

(a) Find the equation of the plane Π_1 that passes through A and contains ℓ_1.

[4 marks]

(b) Find the equation of the plane Π_2 that passes through A and is perpendicular to ℓ_1.

[2 marks]

(c) Find the position vector of the point B where the line ℓ_1 meets the plane Π_2. Hence, or otherwise, find the distance of the point A from the line ℓ_1.

[5 marks]

(d) Find the equation of the plane Π_3 passing through A that is perpendicular to both Π_1 and Π_2.

[4 marks]

7. (a) A complex number z satisfies the equation

$$|z-1+3i| = 2|z-i|.$$

Show that the locus of the point on the Argand diagram associated with z describes a circle. Find the radius and the coordinates of the centre of this circle.

[7 marks]

(b) Find the three roots of the equation $(2z+1)^3 = (z+2i)^3$.

[8 marks]

8. (a) Use the principle of mathematical induction to prove that for every integer $n \ge 1$,

$$\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\cdots+\frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}.$$

[8 marks]

(b) Prove by the principle of mathematical induction that $11^n - 4^n$ is divisible by 7 for every integer $n \ge 1$.

[7 marks]

9. Let Π_1, Π_2 and Π_3 be planes given by the following Cartesian equations, where $k \in \mathbb{R}$.

$$\begin{array}{llllllll} \Pi_1: & -2x & + & ky & + & z & = & -3 \\ \Pi_2: & -x & + & 3y & + & kz & = & -1 \\ \Pi_3: & x & - & 7y & - & 5z & = & 0 \end{array}$$

(a) Find the range of values of k for which the planes Π_1, Π_2 and Π_3 meet at exactly one point.

[5 marks]

(b) Assume that $k = 2$. Find the Cartesian equation describing the line of intersection of the planes Π_1, Π_2 and Π_3.

[5 marks]

(c) Assume that $k = 5$. Find the coordinates of the unique point of intersection of the planes Π_1, Π_2 and Π_3.

[5 marks]

10. The function f is given by $f(x) = \dfrac{x^2 + 2x - 3}{2x^2 - 3x - 5}$.

(a) Determine the equations of the vertical and horizontal asymptotes of the curve $y = f(x)$.

[3 marks]

(b) Show that the curve $y = f(x)$ has no stationary points.

[3 marks]

(c) Find where the curve $y = f(x)$ cuts the coordinate axes.

[2 marks]

(d) On the same diagram, sketch the curve of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

[5 marks]

(e) Hence, deduce that the equation $(x+3)(x-1) = \pm(2x-5)(x+1)$ has two positive real roots and two negative real roots, and mark them on your graph.

[2 marks]