MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2023 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	6th May 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x}(1+x)\sec y\,,$$

given that $y=0$ when $x=0$. **[10 marks]**

2. (a) Let $y = e^{1+\sin^2 x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \sin 2x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos 2x = 0\,.$$

[Hint: Use the identity $2\sin A\cos A = \sin 2A$.] **[5 marks]**

(b) Find the equation of the normal to the curve given by $y = e^{1+\sin^2 x}$ at $x = -\frac{\pi}{4}$. **[5 marks]**

3. (a) Evaluate the integral $\int_1^2 x^3 \ln x \,\mathrm{d}x$. **[4 marks]**

(b) Use the substitution $u = 1+\sqrt{x}$, to evaluate the integral

$$\int_1^9 \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x\,,$$

giving your answer in the form $p\ln 2 + q$, where p and q are positive integers. **[6 marks]**

4. The lines ℓ_1 and ℓ_2 have vector equations

$$\mathbf{r}_1 = \mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} + \alpha\mathbf{j} + 4\mathbf{k}), \quad \text{and}$$
$$\mathbf{r}_2 = -\mathbf{i} - 3\mathbf{j} + \beta\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

respectively.

(a) Find α and β, given that the lines intersect and are perpendicular to each other. Find the point of intersection A of these two lines. **[5 marks]**

(b) The line ℓ_3 has vector equation

$$\mathbf{r}_3 = 5\mathbf{i} - 12\mathbf{j} + \gamma\mathbf{k} + \nu(\delta\mathbf{i} - 11\mathbf{j} + \varepsilon\mathbf{k}).$$

Find γ, δ and ε, given that ℓ_3 passes through A and is perpendicular to the other two lines. **[5 marks]**

5. (a) Consider the two functions $f(x) = \sqrt{x+2}$ and $g(x) = ax + b$, where a and b are positive integers.

(i) Determine the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

(ii) Find a and b if $(f \circ g)(1) = 3$ and $(g \circ f)(2) = 10$. **[4 marks]**

(b) Let $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $x \in \mathbb{R}$. Find the inverse function $f^{-1}(x)$, giving its domain. **[6 marks]**

6. (a) Solve the equation $\sin\theta + \cos 2\theta - \sin 3\theta = 0$ for $0 \le \theta \le 2\pi$.
[Hint: One might need to use a suitable substitution for $\sin\theta - \sin 3\theta$ or for $\sin 3\theta$.] **[4 marks]**

(b) Solve for x the following equations:

(i) $\log_{10}(3x+1) = 4\log_{100} x + 1$.

(ii) $\pi^{x-1} = 2^{x+3}$. **[6 marks]**

7. (a) How many terms of the geometric series

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

must be taken for the sum to get within 0.01% of its sum to infinity? **[5 marks]**

(b) Express

$$\frac{1}{1 - x + x^2 - x^3}$$

in partial fractions, and hence expand the expression in ascending powers of x, giving the first six terms. State the necessary restrictions on the values of x. **[5 marks]**

8. (a) Prove that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. **[5 marks]**

   (b) Hence, express in partial fractions

$$\frac{2x^2-51x+74}{(3x-5)^3+(4-x)^3+(1-2x)^3}.$$

[5 marks]

9. (a) How many different five-letter words can be formed using the letters of the word

BUTTERFLY?

[6 marks]

   (b) What is the probability that a word chosen at random from all the different five-letter words would have the two letters T next to each other? **[4 marks]**

10. (a) A circle $\mathscr{C}_1$ has equation $x^2+y^2-8x-2y+1=0$. Find the radius and the coordinates of the centre of $\mathscr{C}_1$. **[3 marks]**

    (b) The transformation **T** in two dimensions is composed of an anticlockwise rotation of $\pi/2$ radians about the origin followed by a reflection in the x-axis. Write down the matrix representation of **T**. **[4 marks]**

    (c) The transformation **T** is applied on $\mathscr{C}_1$ to obtain $\mathscr{C}_2$. Derive the equation of $\mathscr{C}_2$. **[3 marks]**

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MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2023 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	8^{th} May 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) (i) Use a suitable substitution to find the integral $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x$.

 (ii) Hence, solve the first order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x\ln x}=\frac{1-x}{(\ln x)^2}, \qquad \text{where } x\in\mathbb{R}^+\setminus\{1\},$$

given that $y=e$ when $x=e$. Give your answer in the form $y=f(x)$. **[8 marks]**

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-6\frac{\mathrm{d}y}{\mathrm{d}x}+10y=10e^{2x},$$

given that $y=6$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=10$ when $x=0$. Give your answer in the form $y=f(x)$.
[7 marks]

2. Consider the curve $\mathscr{C}$ with polar equation $r=f(\theta)$ for $0\le\theta\le\frac{\pi}{2}$, where

$$f(\theta)=2\sqrt{\sec\theta\operatorname{cosec}\theta}\,.$$

(a) Derive the Cartesian equation of the curve $\mathscr{C}$ and hence, sketch the curve. **[6 marks]**

(b) On the same sketch, shade the area A bounded between the curve and the half lines $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$. **[2 marks]**

(c) Show that $\sec\theta\operatorname{cosec}\theta=\dfrac{\sec^2\theta}{\tan\theta}$ and hence, determine the value of the area A. **[7 marks]**

3. (a) (i) Evaluate $\displaystyle\int_0^{\pi} x\cos(x/2)\,\mathrm{d}x$.

(ii) Let $I_n = \displaystyle\int x^n \cos ax \,\mathrm{d}x$ for $a > 0$. Show that, for $n \geq 2$,

$$I_n = \frac{1}{a}x^n \sin ax + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}I_{n-2}.$$

(iii) A curve in the xy-plane is defined parametrically in terms of t. Given that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^3 \sin t \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = t^3(1+\cos t),$$

find the arc length of the curve from the point where $t = 0$ to the point where $t = \pi$. [Hint: Use the fact that $1 + \cos A = 2\cos^2(A/2)$.] **[9 marks]**

(b) (i) Evaluate $\displaystyle\int x\ln x\,\mathrm{d}x$ and $\displaystyle\int x(\ln x)^2\,\mathrm{d}x$.

(ii) The region bounded by the curve $y = x^{1/2}\ln x$ and the x-axis between $x = 1$ and $x = e^2$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation. **[6 marks]**

4. (a) In how many different ways can six similar coins be arranged on a 6×6 grid so that no two coins lie in the same row or column? **[5 marks]**

(b) An urn contains ten balls numbered from 1 to 10. A ball chosen at random from the urn is given to Paul, and then another ball chosen at random from the remaining balls in the urn is given to Gertrude. What is the probability that Paul and Gertrude have balls with consecutive numbers? **[5 marks]**

(c) The alphabets of three different languages are composed of the following letters:

$$\text{Language A}: \{a, b, c, d, e, f, g, h\}$$
$$\text{Language B}: \{a, b, c, d, i, j, k, l\}$$
$$\text{Language C}: \{c, d, e, i, k, l, m, n\}$$

(i) Represent this information on a Venn diagram.

(ii) What is the probability that a letter chosen at random from the set of available letters features in exactly two of the alphabets? **[5 marks]**

5. The points A, B and C have position vectors $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$, $\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}$, respectively.
   (a) Find the equation of the plane Π_1 passing through A, B and C. **[3 marks]**
   (b) The point D has position vector $2\mathbf{i}+9\mathbf{j}-6\mathbf{k}$. Verify that D does not lie in the plane Π_1, and write down the vector equation of the line ℓ_1 that passes through D and is perpendicular to Π_1. **[3 marks]**
   (c) Find the position vector of the point E where ℓ_1 and Π_1 intersect. **[3 marks]**
   (d) Let ℓ_2 be the line passing through the points A and E, and let ℓ_3 be the line passing through B and C. Find the point F where the two lines meet. **[4 marks]**
   (e) Does F lie between B and C? Does E lie inside the triangle ABC?
   [Hint: It might be helpful to sketch a two-dimensional diagram showing the relative positions of the points A, B, C and F.] **[2 marks]**

6. (a) Show that the roots of the equation $z^3 = -1$, where $z \in \mathbb{C}$, can be expressed as -1, ω and ω^{-1}. **[3 marks]**
   (b) Verify that $-1+\omega+\omega^{-1} = 1+\omega^2+\omega^{-2} = 0$ and show that

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & \omega & \omega^{-1} \\ -1 & \omega^{-1} & \omega \end{pmatrix}\begin{pmatrix} -1 & -1 & -1 \\ -1 & \omega^{-1} & \omega \\ -1 & \omega & \omega^{-1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

[6 marks]

   (c) Hence, solve the system of simultaneous equations

$$\begin{aligned} x+y+z &= 6 \\ x-\omega^{-1}y-\omega z &= -6 \\ x-\omega y-\omega^{-1}z &= -6, \end{aligned}$$

giving your answer in numerical form (i.e. not in terms of ω). **[6 marks]**

7. The function f is given by $f(x) = \dfrac{2+x}{2+x+x^2}$.

   (a) Find where the curve $y = f(x)$ cuts the coordinate axes. **[2 marks]**
   (b) Find the stationary points of the curve $y = f(x)$ and determine their nature. **[4 marks]**
   (c) Determine the equations of any asymptotes of the curve $y = f(x)$. **[2 marks]**
   (d) Sketch the curve of $y = f(x)$. **[4 marks]**
   (e) Hence, sketch the curve of $y^2 = f(x)$. **[3 marks]**

8. (a) Show that the equation $\sin(3\ln x) = x^5 - 3x^3 - 2$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 2 as a first approximation. Do **TWO** iterations and give your working to four decimal places.
[Note that angles should be taken in radians in this question.] **[7 marks]**

(b) (i) Evaluate the integral $\displaystyle\int_0^1 e^{-x^2+2x}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to four decimal places.

(ii) By using the series expansion of e^t and taking $t = -x^2 + 2x$, show that the series expansion of e^{-x^2+2x}, up to and including the term in x^4, is

$$1 + 2x + x^2 - \frac{2}{3}x^3 - \frac{5}{6}x^4.$$

Use this series expansion to find another estimate of the integral evaluated in part (i).
[8 marks]

9. (a) Prove by the principle of mathematical induction that

$$\sum_{r=1}^{n}(r+1)2^{r-1} = n2^n,$$

for every integer $n \geq 1$. **[6 marks]**

(b) Let $f(x) = (x+1)\ln(1+x) - x$, where $x > -1$.

(i) Use the principle of mathematical induction to prove that for every integer $n \geq 2$, the n^{th} derivative of f is given by

$$f^{(n)}(x) = (-1)^n(n-2)!(1+x)^{-(n-1)}.$$

(ii) Find the first **FOUR** non-zero terms in the Maclaurin series for $f(x)$.

(iii) Find the coefficient of x^n in this series for $n \geq 2$. **[9 marks]**

10. (a) Show that the matrix

$$\mathbf{A} := \begin{pmatrix} 1 & 2 & -3 \\ -k & 1+3k & 3-k \\ 0 & -6 & 5 \end{pmatrix}$$

is singular if and only if $k = -23$. **[3 marks]**

(b) Find $\mathbf{A}^{-1}$ when $k = 3$. **[4 marks]**

(c) Solve for x, y and z the system of equations

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & -1 \\ -kx & + & (1+3k)y & + & (3-k)z & = & 15 \\ & & -6y & + & 5z & = & 2 \end{array}$$

for the cases when $k = 3$ and $k = -23$. Give a geometric interpretation for the solutions obtained. **[8 marks]**