MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2023 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	30th August 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Use a suitable substitution to find the integral $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$. **[3 marks]**

   (b) Hence, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{-y}\ln x}{xy^2},$$

   given that $y = 0$ when $x = e$. **[7 marks]**

2. (a) Let $y = \ln\big(1+\sin(x^2)\big)$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2e^{-y}\big(2x^2\sin(x^2) - \cos(x^2)\big) = 0.$$

   [5 marks]

   (b) A curve has equation $4y^2 - 7xy + 3x^2 - 21 = 0$. Find the equation of the line that is tangent to the curve at the point $(5,2)$. **[5 marks]**

3. The points A, B and C have position vectors $5\mathbf{i}+35\mathbf{j}+6\mathbf{k}$, $8\mathbf{i}+\alpha\mathbf{j}$ and $\beta\mathbf{i}-5\mathbf{j}-4\mathbf{k}$, respectively.

   (a) Find α and β given that A, B and C are collinear. **[4 marks]**

   (b) The point D has position vector $2\mathbf{i}+\gamma\mathbf{j}+2\mathbf{k}$. Find the TWO possible values of γ given that the angle BDC is a right angle. **[3 marks]**

   (c) Find the area of the triangle BDC for one of the values of γ found in part (b). **[3 marks]**

4. The point A has coordinates $(2,5)$ and the line ℓ_1 has equation $2x+4y-14=0$.
   (a) Find the equation of the line ℓ_2 passing through A which is parallel to the line ℓ_1. **[3 marks]**
   (b) Find the equation of the line ℓ_3 passing through A which is perpendicular to the line ℓ_1. **[3 marks]**
   (c) Find the locus of the points which are equidistant from the two lines ℓ_1 and ℓ_2. **[4 marks]**

5. (a) Use integration by parts to find the integral $\int e^{2x} \sin x \; \mathrm{d}x$. **[4 marks]**
   (b) Use the substitution $x = \tan\theta$ to show that:

$$\int \frac{1}{(x^2+1)^2} \, \mathrm{d}x = \int \cos^2\theta \; \mathrm{d}\theta \, .$$

   Hence, evaluate $\displaystyle\int_0^1 \frac{1}{(x^2+1)^2} \, \mathrm{d}x$. **[6 marks]**

6. (a) Express into partial fractions

$$\frac{x^2+3x-2}{x^3-x^2+x-1} \, .$$

[4 marks]

   (b) If α and β are the roots of the quadratic equation

$$(1+i)z^2 - 2iz + (4+i) = 0 \, ,$$

   where $i=\sqrt{-1}$, express $\alpha+\beta$ and $\alpha\beta$ in the form $a+ib$, where a and b are real. Find, in a form not involving α and β, the quadratic equation whose roots are $\alpha+2\beta$ and $2\alpha+\beta$. **[6 marks]**

7. (a) Show that, if x is small enough for its cube and higher powers to be neglected,

$$\sqrt{\frac{1-x}{1+x}} = 1 - x + \frac{x^2}{2} \, .$$

   By putting $x = 1/8$, show that $\sqrt{7} \approx 2\frac{83}{128}$. **[5 marks]**
   (b) The seventh term of an arithmetic progression is 4 and the second, fifth and eleventh terms are consecutive terms in a geometric progression. Find the first term of the arithmetic progression and also the common difference, assuming it is **NOT** zero. What is the common ratio of the geometric progression? **[5 marks]**

8. (a) The Hawaiian alphabet is made up of the seven consonants h, k, l, m, n, p and w, together with the five vowels a, e, i, o and u. Find how many different five-letter words can be formed by using exactly one vowel and four consonants if:
   (i) The letter a must be used in the centre of the word and **NO** letter can be repeated;
   (ii) **NO** letter can be repeated;
   (iii) Any letter can be repeated any number of times. **[6 marks]**

   (b) Find the values of the constants a, b, c and d such that the matrices **P** and **Q** given below are the inverse of each other.

$$\mathbf{P} = \begin{pmatrix} a & 2 & -1 \\ 2 & 2 & 1 \\ 4 & 3 & b \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} -1 & -5 & c \\ 2 & d & -7 \\ -2 & -7 & 6 \end{pmatrix}.$$

[4 marks]

9. (a) Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[3]{5x-1}$ and $g(x) = \frac{1}{5}(x^3+1)$.
   (i) Determine the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$.
   (ii) Hence, find the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$. **[5 marks]**

   (b) The function $f : [0, \pi] \to \mathbb{R}$ is given by $f(x) = 3^{\cos x} + \frac{1}{6}$.
   (i) Explain why f is injective (i.e. $1-1$-function).
   (ii) Find the range of f, i.e. ran(f).
   (iii) Find the inverse function $f^{-1} : \text{ran}(f) \to [0, \pi]$. **[5 marks]**

10. (a) Express $4\cos\theta - 3\sin\theta$ in the form $R\cos(\theta+\phi)$, where $R > 0$ and $\phi \in [0, 2\pi]$, giving the exact value of R and the value of ϕ correct to 3 decimal places. Hence, solve the equation $4\cos\theta - 3\sin\theta = 1$ for $\theta \in [0, 2\pi]$. Give your answer correct to 3 decimal places. **[5 marks]**

   (b) If the expression $ax^4 + bx^3 - x^2 + 2x + 3$ leaves remainder $\sqrt{3}x + 5$ when it is divided by $x^2 - x - 2$, find the values of a and b. Give your answer in surd form. **[5 marks]**

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2023 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	31st August 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Solve the first order linear differential equation

$$(x-1)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=(x-1)e^{-x},$$

given that $y=1$ when $x=0$. Give your answer in the form $y=f(x)$. **[7 marks]**

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\frac{\mathrm{d}y}{\mathrm{d}x}+4y=12x^2+4x,$$

given that $y=\frac{3}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=0$. Give your answer in the form $y=f(x)$.
[8 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $e^{2x}\cos 5x=x^4+5x^2-3$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 0.5 as a first approximation. Do **TWO** iterations and give your working to four decimal places.
[8 marks]

(b) Evaluate $\int_0^1 \sin(x^3-x+1)\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=0.25$. Give your answer to four decimal places. **[7 marks]**

3. (a) By reducing to echelon form, or otherwise, solve the equation

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

[5 marks]

(b) Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **TRACE** of a square matrix $\mathbf{X}$, denoted by tr[$\mathbf{X}$], is defined to be the sum of the entries on the leading diagonal (the diagonal from the upper left to the lower right entries) of $\mathbf{X}$. For example, for the above 2×2 matrix $\mathbf{A}$, one has $\text{tr}[\mathbf{A}] = a + d$.

(i) Find det[$\mathbf{A}$], tr[$\mathbf{A}^2$] and det[$\mathbf{A}^2$] in terms of a, b, c and d.

(ii) Verify that $\det[\mathbf{A}^2] = (\det[\mathbf{A}])^2$ and $\text{tr}[\mathbf{A}^2] = (\text{tr}[\mathbf{A}])^2 - 2\det[\mathbf{A}]$.

(iii) Show that $\mathbf{A}^2 - (\text{tr}[\mathbf{A}])\mathbf{A} + (\det[\mathbf{A}])\mathbf{I} = \mathbf{0}$, where $\mathbf{I}$ is the 2×2-identity matrix and $\mathbf{0}$ is the 2×2-zero matrix. Hence, or otherwise, find $\mathbf{A}$ when $\mathbf{A}^2 = \begin{pmatrix} 1 & 18 \\ -6 & 13 \end{pmatrix}$.

[10 marks]

4. (a) (i) Find the two square roots of $3 - 4i$, giving your answer in the form $x + iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

(ii) Draw these on an Argand diagram, labelling the points A and B.

(iii) Find two possible points C_1 and C_2 such that triangles ABC_1 and ABC_2 are equilateral.

[8 marks]

(b) Use de Moivre's Theorem to show that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

Hence, by substituting $t = \tan\theta$, solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$, giving your answer correct to three decimal places. **[7 marks]**

5. (a) (i) How many subsets of the set of integers $\{1, 2, \ldots, 8\}$ can be formed (include also the empty set and the whole set itself)?

(ii) How many of these subsets do **NOT** contain two consecutive integers?

(iii) If one of the subsets is chosen at random, what is the probability that it contains two successive integers? **[12 marks]**

(b) If the two events A and B are independent, show that:

$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A)\text{P}(B).$$

[3 marks]

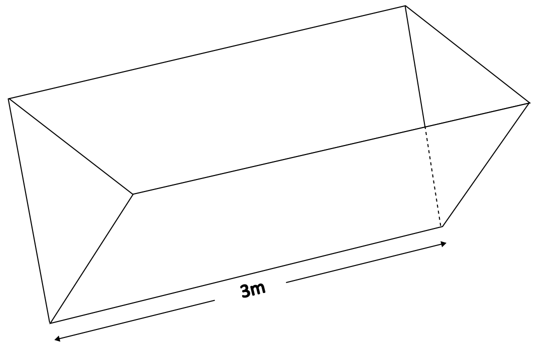
6. (a) (i) Let $I_n = \displaystyle\int \frac{x^n}{\sqrt{1+x}}\,dx$. Show that, for $n \geq 1$,

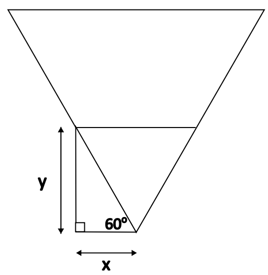
$$(2n+1)I_n = 2x^n\sqrt{1+x} - 2nI_{n-1}.$$

(ii) Evaluate I_0, I_1 and I_2.

(iii) The region bounded by the curve $y = \dfrac{x^{3/2}}{(1+x)^{1/4}}$ and the x-axis between $x = 0$ and $x = 3$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation. **[9 marks]**

(b) A water trough in a farm is shaped like a triangular prism. It is 3 meters long and its cross section is an equilateral triangle (see diagram, where y is the height of the water in the trough). The trough is filling up at a rate of 6 $cm^3\ s^{-1}$.





Find the rate of change of the surface area of the water in contact with the air, when the depth of the water in the trough is 20 cm. **[6 marks]**

7. Let $f(\theta) = \dfrac{3\cos\theta}{\cos 2\theta}$ for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, and let the curve $\mathscr{C}_1$ have polar equation $r = f(\theta)$.

(a) Which are the values of θ in the given domain for which $f(\theta)$ is **NOT** defined? **[3 marks]**

(b) Obtain the Cartesian equation of the curve $\mathscr{C}_1$. **[4 marks]**

(c) Hence, sketch the curve $\mathscr{C}_1$, showing clearly any asymptotes. **[4 marks]**

(d) The curve $\mathscr{C}_2$ has polar equation $r = 3\cos\theta$ for $0 \leq \theta \leq \pi$. Find the polar coordinates of the common point on the curves $\mathscr{C}_1$ and $\mathscr{C}_2$. **[4 marks]**

8. (a) Let $f(x) = e^x \sin x$.

(i) Show that $f''(x) = 2(f'(x) - f(x))$.

(ii) Hence, or otherwise, find the Maclaurin series for $f(x)$, up to and including, the term in x^5. **[7 marks]**

(b) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

[8 marks]

9. The function f is given by $f(x) = \dfrac{x^2 - 3x - 4}{2x^2 + 7x + 3}$.
   (a) Find where the curve $y = f(x)$ cuts the coordinate axes. **[2 marks]**
   (b) Show that the curve $y = f(x)$ does **NOT** have any stationary points. **[3 marks]**
   (c) Determine the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. **[2 marks]**
   (d) Find where the curve cuts the line representing the horizontal asymptote. **[2 marks]**
   (e) Sketch the curve of $y = f(x)$ and hence, sketch also the curve of $y = |f(x)|$. **[6 marks]**

10. The lines ℓ_1 and ℓ_2 have vector equations

$$\mathbf{r}_1 = 3\mathbf{i} - 11\mathbf{j} - 10\mathbf{k} + \lambda(\mathbf{i} - 7\mathbf{j} - 9\mathbf{k}), \text{ and}$$
$$\mathbf{r}_2 = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

respectively.
   (a) Show that the lines intersect. Find the position vector of their point of intersection A, and the equation of the plane Π_1 in which they both lie. **[5 marks]**
   (b) The line ℓ_3 has vector equation

$$\mathbf{r}_3 = 2\mathbf{i} + 6\mathbf{j} + 19\mathbf{k} + \nu(\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}).$$

   Show that ℓ_3 also lies in Π_1. **[4 marks]**
   (c) The point B has position vector $3\mathbf{i} + \mathbf{j} + 8\mathbf{k}$. Verify that B does **NOT** lie in the plane Π_1, and find the equation of the plane Π_2 containing B and ℓ_1. Find the acute angle between the planes Π_1 and Π_2. **[6 marks]**