MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

INTERMEDIATE MATRICULATION LEVEL
2020 SECOND SESSION

SUBJECT:	**Pure Mathematics**
DATE:	12th December 2020
TIME:	16:00 to 19:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.

1. (a) Solve the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy\,,$$

given that $y = 2$ when $x = 0$.

[5 marks]

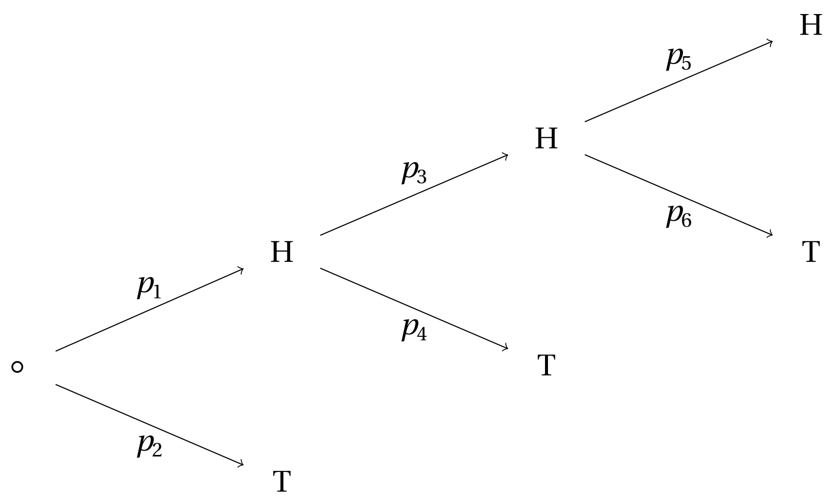
(b) Find the area between the two curves $y = x^2$ and $y = 2x - x^2$.

[5 marks]

2. Consider the graph of $f(x) = \frac{1}{3}x^3 + 2x^2 - 5x$. There is a maximum point at M, and a minimum point at N on the graph.
(a) Find $f'(x)$.
(b) Find the coordinates of M and N.
(c) Find the coordinates of the points where the graph cuts the x-axis.
(d) Sketch the graph of f from $x = -9$ to $x = 4$.
(e) Find the equation of the tangent line ℓ to the curve of f at the point $(3, 12)$ in the form $y = mx + c$.
[Give the coordinates correct to one place of decimal.]

[2, 2, 2, 2, 2 marks]

3. You play the following game. You flip a fair coin. If it comes up tails (T), you get 2 points and the coin is not flipped again. If the coin comes up heads (H), you get only 1 point but you can flip the coin again. You can flip the coin at most three times in one turn. If you flip the coin multiple times in one turn, you add up the points. This is represented by the following tree diagram, where $p_1, \ldots, p_6$ are probabilities.



(a) Find the probabilities $p_1, \ldots, p_6$.

[1 mark]

(b) What is the probability that you will get exactly 3 points in one turn?

[4 marks]

(c) What is the probability that you will get either 2 points or 4 points?

[2 marks]

(d) If the total score is 3, you win €10. If the total score is 2 or 4, you get nothing. If you play the game twice, what is the probability that you win exactly €10?

[3 marks]

4. Consider $f(x) = x^3 - 3x^2 + 4$.

(a) Show that $(x-2)$ is a factor of $f(x)$.

[2 marks]

(b) Hence, factorise completely $f(x)$.

[2 marks]

(c) Express $\dfrac{3}{x^3 - 3x^2 + 4}$ in partial fractions.

[3 marks]

(d) Hence, find $\displaystyle\int \frac{3}{x^3 - 3x^2 + 4}\,\mathrm{d}x$.

[3 marks]

5. (a) Consider the arithmetic series $2+5+8+\ldots$.
   (i) Find an expression for S_n, the sum of the first n terms.
   (ii) Find the value of n for which $S_n = 1365$.

[2, 3 marks]

(b) The sum of the first two terms of a geometric sequence is 15 and the sum to infinity is 27. If the terms of the sequence are all positive, find the value of:
   (i) the common ratio;
   (ii) the first term.

[3, 2 marks]

6. (a) Let $f(x) = \sqrt{\dfrac{1}{x^2} - 2}$.
   (i) Find the set A of real values of x for which $f(x)$ is **real** and **finite**.
   (ii) Find the range of the function f with domain A.

[3, 1 marks]

(b) Find the range of values of k such that

$$k(x+1) \leq x^2$$

for all real x.

[6 marks]

7. A stone is tossed in the calm water of a lake. A circular ripple develops on the surface of the water which starts to increase in size. The radius of the circular ripple is given by the formula

$$r = \frac{3(1+t^3)}{9+t^3},$$

where r is the radius in centimetres of the circular ripple and t is the time in seconds after the stone touches the water.

(a) Find t when $r = 7/3$.

[2 marks]

(b) Find a simplified expression, in terms of t, for the rate of change of the radius.

[2 marks]

(c) Find the rate of change of the area of the circular ripple when $r = 7/3$.

[3 marks]

(d) Find the value of t, $t > 0$, when the rate of change of the radius reaches a maximum.

[3 marks]

8. (a) The quadratic equation $ax^2 + bx + c = 0$, where a, b and c are integers, has roots α and β. Find a, b and c given that $\alpha - \beta = 2$ and $\alpha^2 - \beta^2 = 3$.

[6 marks]

(b) The angle ϕ satisfies the equation $3\sin^2\phi - 7\sin\phi + 5 = 3\cos^2\phi$, where ϕ is in degrees. Find **all** possible values of ϕ satisfying $0° \leq \phi \leq 90°$.

[4 marks]

9. Let the matrix **T** be given by $\mathbf{T} = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$, and let $P(a, 4a)$ be a point on the line with equation $y = 7x - 3a$, where $a \neq 0$.

(a) Find the coordinates of P', if $P'(x', y')$ is the image of $P(a, 4a)$ under the transformation **T**.

[2 marks]

(b) The transformation **T** maps the line with equation $y = 7x - 3a$ onto another straight line with equation $y = kx - 3a$. Use your answer to part (a) to find the equation of this straight line.

[2 marks]

(c) Let $Q(a, (m-3)a)$ be a point on the line with equation $y = mx - 3a$. Find the coordinates of Q', if $Q'(x', y')$ is the image of $Q(a, (m-3)a)$ under the transformation **T**.

[2 marks]

(d) Given that the transformation **T** maps two lines of the form $y = mx - 3a$ onto themselves, use your answer to part (c), or otherwise, to find the **two** possible values of m.

[4 marks]

10. (a) Find the exact value of x satisfying the equation

$$(3^x)(4^{2x+1}) = 6^{x+2}.$$

Give your answer in the form $\dfrac{\ln a}{\ln b}$, where a and b are integers.

[6 marks]

(b) When the polynomial $P(x) = 6x^3 + px^2 + qx - 1$ is divided by $(x - 1)$ the remainder is -2. When $P(x)$ is divided by $(2x - 1)$ the remainder is $\dfrac{3}{2}$. Find the value of p and q.

[4 marks]